Physics 11b
Lecture #3

Electric Flux
Gauss’s Law
What We Did Last Time

- Introduced electric field \( \mathbf{E} \) by \( \mathbf{F}_Q = Q \cdot \mathbf{E}(\mathbf{r}) \)
- Field lines and the rules
  - From a positive charge to a negative charge
  - No splitting, merging, or crossing
  - Number of field lines \( \propto \) amount of charge
  - Density \( \propto \) \( \mathbf{E} \) field
- Was about to define electric flux
  - We continue from there…
Today’s Goals

- Define electric flux $\Phi_E$
  - Start from the number of field lines
- Introduce Gauss’s Law
  - Useful tool for many E&M problems
- Apply Gauss’s Law to a few examples
  - Spherical charge distribution
  - Infinite sheet of charge
- Discuss basic rules of conductors
  - Derive them using Gauss’s Law
Consider a flow of water

- The water velocity is described by

\[ \mathbf{v}(x, y, z) \equiv v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \equiv (v_x, v_y, v_z) \]

Immerse a tiny wire loop

- Area of the loop is \( A \)
- Unit vector perpendicular to the loop is \( \hat{n} \)

Define the loop area vector as \( \mathbf{A} \equiv A \hat{n} \)

- It represents the size and the orientation of the loop
- The loop is so small that the shape is irrelevant

Q: how much water will flow through the loop?

- Let’s call it “flux” \( \Phi_v \)
Water Flux

- It depends on how the loop is oriented w.r.t. the flow
  - Assuming constant velocity and a flat loop:
    - If $A \parallel v \rightarrow \Phi_v = 0$
    - If $A \perp v \rightarrow \Phi_v = A v$
    - If $A$ and $v$ makes angle $\theta \rightarrow \Phi_v = A v \cos \theta$

- Generalize:
  
\[ \Phi_v = \int v \cdot dA \]

Integrate over the area of the loop
Imagine a small area in an electric field
- Let’s call the area $A$
- It could be any shape, any angle

How many field lines run through this area?

Remember: density of field lines $\propto E$
- So we are talking about “flux” of $E$
- Math is identical to the water flow case

$$\Phi_E = \int E \cdot dA$$
We now define electric flux as \( \Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} \)

- \( S \) is a surface area, \( d\mathbf{A} = dA \mathbf{n} \)
- \( \Phi_E \) is proportional to the number of field lines going thru \( S \)

- **Unit** of \( \Phi_E \) is Newton·m²/Coulomb (N·m²/C)

- **NB**: sign of flux depends on the direction of \( \mathbf{n} \)
  - That is, you must define which side of \( S \) is “positive”

\[ S = 1 \, \text{m}^2 \quad \Phi_E = -1 \, \text{Nm}^2/\text{C} \quad E = 1 \, \text{N/C} \]

\[ S = 1 \, \text{m}^2 \quad \Phi_E = 1 \, \text{Nm}^2/\text{C} \]
Which Way is $d\mathbf{A}$?

- Defined unambiguously only for a closed surface
  - i.e., a surface that wraps around a volume completely
  - At any point in space, $d\mathbf{A}$ is perpendicular to the surface
  - It points towards the “outside” of the surface

- In other words: flux $\Phi_\mathbf{E}$ (or $\Phi_\mathbf{v}$) is positive if the net flow is coming out of the volume
Consider a sphere of radius $r$ around a charge $q$

- $n$ always points outward
- We know $E$ and $n$ are parallel
- We also know $E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$

$$\Phi_E = \int_{\text{sphere}} E \cdot dA = 4\pi r^2 \times E = \frac{q}{\varepsilon_0}$$

This is hardly surprising

- $\Phi_E$ should be proportional to the number of field lines coming out of charge $q$, which should be proportional to $q$
- We just didn’t know that the constant was $1/\varepsilon_0$
Consider a sphere in a uniform $\mathbf{E}$ field

- Take polar coordinates $(\theta, \phi)$ relative to the direction of $\mathbf{E}$

\[
\Phi_E = \iint E \cos \theta r^2 \sin \theta d\theta d\phi
\]

\[
= 2\pi Er^2 \int_0^\pi \cos \theta \sin \theta d\theta
\]

\[
= 0
\]

Again hardly surprising

- Incoming $\mathbf{E}$ on one side is balanced by outgoing $\mathbf{E}$
- We know field lines never disappear without a charge
Gauss’s Law

- Net flux through a closed surface is given by the net charge inside the surface by

\[ \Phi_E = \oint E \cdot dA = \frac{q_{in}}{\varepsilon_0} \]

- This seems natural, from what we’ve found so far
- Formal proof is found in textbook 24.5

- The law connects charge and field in yet another way
  - Coulomb’s law did it one way – and they are consistent

- It’s more useful than it looks
  - Let’s investigate
Problem: Calculate the electric field (everywhere in space) due to a uniformly-charged sphere

Solid sphere of radius $R$
Constant charge density

Solution #1

I know the $E$ due to a point charge $dq$ by Coulomb’s Law
I know how to integrate
Solution #2

Why would I ever do an integral is somebody (Gauss) already did it for me?

First, consider a sphere $S_1$ outside $R$

Apply Gauss’s Law to $S_1$

$$
\Phi_E = \oint_{S_1} \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}
$$

Because of symmetry, $\mathbf{E}$ should be same size and parallel to $d\mathbf{A}$ everywhere on $S_1$

$$
\oint_{S_1} \mathbf{E} \cdot d\mathbf{A} = E \cdot 4\pi r^2
$$

Combine

$$
E = \frac{Q}{4\pi \varepsilon_0 r^2}
$$
Apply Gauss’s Law

- First, consider a sphere $S_2$ inside $R$
  - Apply Gauss’s Law to $S_2$
    \[
    \Phi_E = \oint_{S_1} \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\varepsilon_0}
    \]
  - Difference: $q_{\text{in}}$ is only a fraction of $Q$ that is inside $S_2$
    \[
    q_{\text{in}} = Q \frac{r^3}{R^3}
    \]
    - Ratio of volumes = (ratio of radii)$^3$
  - Same symmetry argument gives
    \[
    \oint_{S_2} \mathbf{E} \cdot d\mathbf{A} = E \cdot 4\pi r^2
    \]
    - Combine
    \[
    E = \frac{Qr}{4\pi\varepsilon_0 R^3}
    \]
One point remains before we can get full credit
- We were asked to determine the electric field \( \Rightarrow \) a vector
- We need to specify the direction!

Which way is \( \mathbf{E} \)?
- Problem is spherically symmetric
  - \( \mathbf{E} \) must point radially (in or out)

Complete solution:

\[
\mathbf{E} = \begin{cases} 
\frac{Q}{4\pi \varepsilon_0 r^2} \hat{r} & \text{for } r \geq R \\
\frac{Qr}{4\pi \varepsilon_0 R^3} \hat{r} & \text{for } r < R 
\end{cases}
\]
Checklist for E&M Problems

- Read the problem
- Look for symmetries:
  - Which coordinate system works best?
  - What cancels out?
  - Which way the vectors should be?
- Look for ways to avoid integration
- Turn the math crank
- Write down the complete solution (magnitudes and directions for all the different regions)
- Read the problem again – did you answer what is asked?
- Box the solution: your TF will be grateful
Problem: Calculate the electric field at a distance $z$ from a positively charged infinite plane.

- Surface charge density: $\sigma = \frac{\text{charge}}{\text{area}}$

Use Gauss again:
- Which surface to use?
  - What symmetry do we have?
- Consider the cylinder ➔
  - Area $A$ and height $z$

- $E$ field must be vertical
  - How do we know that?
Infinite Sheet of Charge

- Now the total flux $\Phi_{\text{total}} = \Phi_{\text{top}} + \Phi_{\text{side}} + \Phi_{\text{bottom}}$
  - Side is parallel to $\mathbf{E} \Rightarrow$ No flux
  - Top and bottom are symmetric $\Rightarrow$ Same flux

$$\Phi_{\text{total}} = 2\Phi_{\text{top}} = 2AE$$
$$q_{\text{in}} = \sigma A$$

$2AE = \frac{\sigma A}{\varepsilon_0}$
$$E = \frac{\sigma}{2\varepsilon_0}$$

- Don’t forget the direction!

$$\mathbf{E} = \begin{cases} 
    +\frac{\sigma}{2\varepsilon_0} \hat{z} & \text{for } z > 0 \\
    -\frac{\sigma}{2\varepsilon_0} \hat{z} & \text{for } z < 0 
\end{cases}$$

The result is worth remembering:

Infinite sheet of charge produces uniform $\mathbf{E}$ field of $\sigma/2\varepsilon_0$ above and below it
Conductor

- A conductor conducts electric current
  - Conductive metal contains freely-movable electrons
- We won’t deal with current until later
  - What we are doing is “electrostatics” = no moving charges
  - When there is no charge movement, the conductor is in electrostatic equilibrium
- Immediately obvious: there shouldn’t be any E field in the conductor in electrostatic equilibrium
  - Otherwise the free electrons would be moving
  - There are less obvious but interesting rules
Free electrons are negative, but the atoms in the conductors are charge neutral

- So the “core” of the atom must be a positive ion
- Ions are fixed, electrons move
- Their charges usually cancel

What if we took away (or add) some electrons?

- There will be net positive (or negative) charge
- Where does that go?

Answer: on the surface of the conductor
Suppose there is a net charge $+q$ somewhere inside a conductor in electrostatic equilibrium.

Imagine a sphere around the charge and apply Gauss's law:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0} \neq 0$$

For the integral to be non-zero, $\mathbf{E}$ must be non-zero somewhere on the surface of $S$.

But $\mathbf{E}$ is zero inside the conductor!

Rule #2: there is no net charge anywhere inside a conductor in electrostatic equilibrium.
Surface Charge and $E$

- Charge can only be on the surface
  - Let’s call the charge density $\sigma = \frac{Q}{\text{area}}$
  - NB: $\sigma$ may not be constant for the whole surface
- Apply Gauss on a cylinder sticking out of the metal surface
  - $\Phi_{\text{total}} = \Phi_{\text{top}} + \Phi_{\text{side}} + \Phi_{\text{bottom}}$
    
    $= 0$

  - $\Phi_{\text{total}} = \Phi_{\text{top}} = AE$
  - $q_{\text{in}} = \sigma A$

- Rule #3: $E$ field just outside a conductor in electrostatic equilibrium is perpendicular to the surface and the magnitude is $\frac{\sigma}{\varepsilon_0}$
Defined electric flux \( \Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} \)

- \( \propto \) number of field lines through the surface

Gauss’s Law \( \Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} \)

- Very useful for solving \( \mathbf{E} \) field problems

Applied it on spherical charge distribution and infinite sheet

- Infinite sheet generates uniform \( \mathbf{E} \) field \( E = \frac{\sigma}{2\varepsilon_0} \)

Also used on conductors in electrostatic equilibrium

- Charge (if any) lives on the surface