Electric Potential

Textbook Chapter 25
What We Did Last Time

- Defined electric flux \( \Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} \)
  - \( \propto \) number of field lines through the surface

- Gauss’s Law \( \Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} \)
  - Very useful for solving \( \mathbf{E} \) field problems
  - Applied it on spherical charge distribution and infinite sheet
    - Infinite sheet generates uniform \( \mathbf{E} \) field \( E = \frac{\sigma}{2\varepsilon_0} \)
  - Also used on conductors in electrostatic equilibrium
    - Charge (if any) lives on the surface
Today’s Goals

- Introduce electric potential $V$
  - How much work is necessary to move a charge in an electric field?
  - Similar to potential energy in 11a
- Calculate the electric potential generated by
  - a point charge
  - multiple point charges
  - continuous charge distribution
- Learn to go from $V$ to $E$ and $E$ to $V$
  - Not so difficult, but takes a little practice
How much work is necessary to lift an object of mass \( m \) by a height \( h \)?

- Work = force \( F \) \( \times \) distance

 Same work is needed if you go up the slope because gravity is a conservative force.

Object at height \( h \) has a potential energy \( mgh \).
Electric Potential Energy

- Consider moving a charge \( +q \) against \( \mathbf{E} \) field
  - How much work \( W \) is necessary?
    \[
    F = qE \quad \Rightarrow \quad W = Fx = qEx
    \]
- What if the movement is at an angle?
  - What matters is the component of \( \mathbf{F} \) parallel to the movement
  - Let’s call the movement \( \mathbf{s} \)
    \[
    F_{\parallel} = -F \cos \theta \quad \Rightarrow \quad W = F \cos \theta \times s = -\mathbf{F} \cdot \mathbf{s} = -q\mathbf{E} \cdot \mathbf{s} = qEx
    \]
Electric Potential Energy

- Make the movement \( s \) very small \( \Rightarrow \) call it \( ds \)
  - (Very small) work \( dW \) is \( dW = -qE \cdot ds \)
- We can now integrate this to calculate work needed for any movement
  \[
  W = -q \int_{A}^{B} E \cdot ds
  \]
- \( W \) doesn’t depend on exactly how you go from \( A \) to \( B \)
  - Any alternative path will cost you exactly the same work
  - In other words, electric force is conservative

How do we know that?
Is Electric Force Conservative?

- Electric forces between any objects are sum of the forces between constituent particles (electrons & protons)
  - The latter is Coulomb:
    \[
    F = \frac{1}{4\pi\varepsilon_0} \frac{qq'}{r^3} \hat{r}
    \]
  - Besides constant and sign, it’s same as gravity
  - Gravity is a conservative force ➔ Electric force must be!

- We’ll see this is true (in a wayward fashion) later
  - Let us take it for granted now…
Electric Potential

- Electric force is conservative
  - *How* you get from *A* to *B* is irrelevant
  \[
  W_{A\to B} = -q\int_A^B \mathbf{E} \cdot d\mathbf{s} = -q\int_A^B \mathbf{E} \cdot d\mathbf{s}'
  \]
  - *W* is a function of positions *A* and *B*
  - *W* is also proportional to *q*

- We can define a function *V(x, y, z)* so that
  \[
  W_{A\to B} = q\left(V(x_B, y_B, z_B) - V(x_A, y_A, z_A)\right)
  \]
  - *V* is called the *electric potential*
  - *qV* is the *electric potential energy* of the charge *q*

Corresponds to the height *h* for gravity
Electric Potential

A charge $q$ moving from $A$ to $B$ “climbs up” the electric potential by $\Delta V = V(B) - V(A)$

- This takes work $W = q\Delta V$
  - The charge gains potential energy $q\Delta V$

$$W_{A\rightarrow B} = -q\int_{A}^{B} \mathbf{E} \cdot d\mathbf{s} \quad \rightarrow \quad \Delta V = -\int_{A}^{B} \mathbf{E} \cdot d\mathbf{s}$$

- Only the potential difference $\Delta V$ can be defined or measured
  - Absolute value at any point (e.g. $V = 0$) has no meaning
  - You can define $V = 0$ in any convenient way
Units

- Work = energy = charge x potential

\[
[potential] = \frac{[energy]}{[charge]} = \frac{\text{Joule}}{\text{Coulomb}} = \text{Volt}
\]

- Potential is also electric field x distance

\[
[field] = \frac{[potential]}{[distance]} = \frac{\text{Volt}}{\text{meter}}
\]

- I told you Newton/Coulomb earlier
  - It’s the same, but V/m is more commonly used

\[\Delta V = -\int_A^B \mathbf{E} \cdot ds\]
Uniform Field Example

- Connect a pair of parallel metal plates to a battery
  - “Positive” plate has a $\Delta V$ higher potential than the “negative”
  - $E$ field between the plates is $E = \frac{\Delta V}{d}$
- Charges $+q$ and $-q$ feel the force $qE = q\frac{\Delta V}{d}$
  - $+q$ toward the negative plate, $-q$ toward the positive plate
- If you let the $+q$ charge go from positive plate, it will “drop” on the negative plate
  - Kinetic energy when it hits the plate $= q\Delta V$
Point Charge Example

- A point charge $+q$ creates electric field
- Let’s calculate $\Delta V$ between $A$ and $B$

$$\Delta V = - \int_A^B \mathbf{E} \cdot d\mathbf{s} = - \int_A^B E \cos \theta ds$$

radial component of $d\mathbf{s}$

$$\Delta V = - \int_{r_A}^{r_B} E dr = - \frac{q}{4\pi \varepsilon_0} \int_{r_A}^{r_B} \frac{1}{r^2} dr = - \frac{q}{4\pi \varepsilon_0} \left[ -\frac{1}{r} \right]_{r_A}^{r_B}$$

- We find $V(r) = \frac{1}{4\pi \varepsilon_0} \frac{q}{r}$ works

Electric potential produced by a point charge
Don’t Get Confused

$E(r) = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r}$

- $E$ is a vector field $\leftrightarrow$ $V$ is a scalar field
- $E$ goes with $1/r^2$ $\leftrightarrow$ $V$ goes with just $1/r$

- Visually:
  - $E$ is expressed either by arrows or by field lines
  - $V$ is expressed by equipotential surfaces
    - Like lines of equal height on maps, but in 3-dim.
Equipotential Surfaces

- They are really spherical shells
- But much easier to visualize in 2-dim.

What happens here?
Let’s add more charges in the system

- \( N \) charges of \( q_i \) each \( (i = 1, 2, \ldots N) \) at positions \( r_i \)
- We know \( E \) just adds up

\[
\Delta V = -\int_{A}^{B} E \cdot ds = -\int_{A}^{B} \sum_{i} E_i \cdot ds
\]

\( V \) must also just add up

\[
V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i} \frac{q_i}{|r - r_i|}
\]
Dipole Example

- Two charges $+q$ and $-q$ at $x = +d/2$ and $-d/2$
- Find potential $V(x,y,z)$ at any point $(x,y,z)$
- Distances from the charges:
  \[ r_1 = \sqrt{(x-d/2)^2 + y^2 + z^2} \]
  \[ r_2 = \sqrt{(x+d/2)^2 + y^2 + z^2} \]

\[
V = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{\sqrt{(x-d/2)^2 + y^2 + z^2}} + \frac{-q}{\sqrt{(x+d/2)^2 + y^2 + z^2}} \right)
\]

- OK, it wasn’t pretty, but at least we didn’t have to add vectors $\rightarrow V$ is often easier to calculate than $E$
- Can we calculate $E$ out of $V$?
Field and Potential

- We get $V$ by line-integrating $E \rightarrow$ What’s reverse?
  - Consider small movement $\Delta x$ in the $x$-direction
    - Potential increases $V \rightarrow V + \Delta V$
      \[
      \Delta V = -\mathbf{E} \cdot \Delta x \hat{x} = -E_x \Delta x
      \]
  - Make $\Delta x$ really small $\rightarrow 0$
    \[
    \lim_{\Delta x \rightarrow 0} \frac{\Delta V}{\Delta x} = \frac{\partial V}{\partial x} = -E_x
    \]
    Same for $y$ and $z$

- $E$ is (negative) derivative of $V$
  \[
  \mathbf{E} = (E_x, E_y, E_z) = -\left( \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)
  \]
  Known as the gradient of $V$
Dipole Example

- We’ve got $V \rightarrow$ Let’s do $E$

\[ V = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{\sqrt{(x-d/2)^2 + y^2 + z^2}} + \frac{-q}{\sqrt{(x+d/2)^2 + y^2 + z^2}} \right) \]

- Let me just do $E_x$

\[ E_x = -\frac{\partial V}{\partial x} = -\frac{1}{4\pi\varepsilon_0} \left( -\frac{1}{2} \frac{2q(x-d/2)}{((x-d/2)^2 + y^2 + z^2)^{3/2}} + \frac{1}{2} \frac{2q(x+d/2)}{((x+d/2)^2 + y^2 + z^2)^{3/2}} \right) \]

\[ = \frac{q}{4\pi\varepsilon_0} \left( \frac{x-d/2}{((x-d/2)^2 + y^2 + z^2)^{3/2}} - \frac{x+d/2}{((x+d/2)^2 + y^2 + z^2)^{3/2}} \right) \]
Dipole Field

- A dipole in two perspectives

- Equipotential and field lines are perpendicular to each other
Continuous Charge Distribution

- We can extend from multiple charge to continuous charge distribution

\[ V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} \rightarrow V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r} \]

- Let’s try on the charged rod example from Lecture #2

\[ r = \sqrt{x^2 + R^2} \]

\[ V(0, R, 0) = \frac{1}{4\pi\varepsilon_0} \int_{-\ell/2}^{\ell/2} \frac{\frac{Q}{l} \, dx}{\sqrt{x^2 + R^2}} \]

Can’t do this integration? Look it up!
**Charged Rod**

- **Formula from textbook**

\[
\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})
\]

**Equation**

\[
V(0, R, 0) = \frac{1}{4\pi \varepsilon_0} \int_{-\ell/2}^{\ell/2} \frac{Q}{\ell} \frac{dx}{\sqrt{x^2 + R^2}}
\]

\[
= \frac{Q}{4\pi \varepsilon_0 \ell} \left[ \ln \left( \frac{\ell}{2} + \sqrt{\frac{\ell^2}{4} + R^2} \right) - \ln \left( -\frac{\ell}{2} + \sqrt{\frac{\ell^2}{4} + R^2} \right) \right]
\]
Summary

- Introduced **electric potential** $V$
  - Work required to move a charge $q$ from $A$ to $B$ is
    \[ W_{A\to B} = -q \int_A^B \mathbf{E} \cdot d\mathbf{s} = q \Delta V \]
  - $\mathbf{E}$ field is negative gradient of $V$
    \[ \mathbf{E} = (E_x, E_y, E_z) = -\left( \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) \]
- A point charge $q$ generates potential
  - For multiple and continuous charge,
    \[ V(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} \quad V(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{dq}{r} \]