Capacitance

S&J Chapter 26
What We Did Last Time

- Electric potential due to continuous charge distributions
  - Use \( V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r} \)
  - Electric field/potential due to spherical charge distribution
    - Looks like a point charge from outside
    - Zero inside
- Discussed conductors
  - Electric shielding (Faraday cage)
- Millikan’s oil-drop experiment
Define capacitance
- Two conductors with electric charge → What is the potential difference between them?
- Study parallel-plate capacitor
  - Will also do a cylindrical one

Combination of capacitors → Rules for “additions”
- Depend on the configuration: parallel and serial

Discuss energy stored in a capacitor
- And where the energy is
Potential Between Conductors

- We’ve learned:
  - Electric potential $V$ of a contiguous conductor is constant
  - Electric field $E$ exists only outside conductors
  - Surface charge $\sigma$ of a conductor is proportional to $E$ just outside the surface

- Consider two charged conductors
  - What’s the $\Delta V = V_1 - V_2$ between them?
Capacitance

- $\Delta V$ must be proportional to $q$
  \[ \Delta V \propto q \]

- Define the capacitance $C$ between the two conductors:
  \[ C \equiv \frac{q}{\Delta V} \]
  or
  \[ q = C \Delta V \]

- It's the amount of electric charge required to produce unit difference in electric potential.

- Unit: Farad = \[ \frac{\text{Coulomb}}{\text{Volt}} \]

- Too big for practical use → Use $\mu F = 10^{-6} F$, pF = $10^{-12} F$
Parallel Plate Capacitor

- Two metallic plates placed close to each other
  - Most common form of capacitors
  - Area $A$, spacing $d$
- Electric field $E$ between the plate is uniform
  - Not at the edge, but that’s a small effect if $d$ is small compared with the size of the plates
- From Lecture #3, we know the relation between $E$ and the charge density

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$
$$\Delta V = Ed = \frac{Qd}{\varepsilon_0 A}$$

$$C = \frac{Q}{\Delta V} = \frac{\varepsilon_0 A}{d}$$

Remember this one?
Charge can only be on the surface
- Let’s call the charge density $\sigma = Q/\text{area}$
- NB: $\sigma$ may not be constant for the whole surface

Apply Gauss on a cylinder sticking out of the metal surface
- $\Phi_{\text{total}} = \Phi_{\text{top}} + \Phi_{\text{side}} + \Phi_{\text{bottom}}$
- $\Phi_{\text{total}} = \Phi_{\text{top}} = AE$
- $q_{\text{in}} = \sigma A$
- $E = \frac{\sigma}{\varepsilon_0}$

Rule #3: $E$ field just outside a conductor in electrostatic equilibrium is perpendicular to the surface and the magnitude is $\sigma/\varepsilon_0$
Parallel Plate Capacitor

- Capacitance of a parallel-plate capacitor
  \[ C = \frac{Q}{\Delta V} = \frac{\varepsilon_0 A}{d} \]
  - Proportional to the area
  - Inversely prop. to the gap

- Does it make sense?
  - Capacitor with larger \( A \) should hold more charge, because the \( E \) field goes with the area density of the charge
  - Capacitor with larger \( d \) should have more potential \( \Delta V \) across it, because it’s \( E \) times the distance \( d \)

- To make a “large” capacitor, you need large plates held together very closely
Cylindrical Capacitor

- Roll up a parallel plate capacitor into cylinders
  - To save space...
- Use Gauss’s Law
  - Imagine a cylindrical surface of radius $r$
  - Top and bottom parallel to $E$ → Ignore
    \[ \Phi_{\text{total}} = \Phi_{\text{side}} = E \times 2\pi r \ell \]
- This must equal to $Q/\varepsilon_0$

\[ 2\pi Er\ell = \frac{Q}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q}{2\pi \varepsilon_0 r \ell} \]

Weaker at larger $r$
Cylindrical Capacitor

- Integrate $E$ to get $\Delta V$
- Note $E$ is parallel to $r$

$$\Delta V = \int_{a}^{b} E \, dr = \int_{a}^{b} \frac{Q \, dr}{2\pi \varepsilon_0 r \ell} = \frac{Q}{2\pi \varepsilon_0 \ell} [\ln r]_{a}^{b}$$

$$= \frac{Q}{2\pi \varepsilon_0 \ell} \ln \left( \frac{b}{a} \right)$$

$$C = \frac{Q}{\Delta V} = \frac{2\pi \varepsilon_0 \ell}{\ln(b/a)}$$

Does this make sense?
Cylindrical Capacitor

- We’ve found \( C = \frac{Q}{\Delta V} = \frac{2\pi \varepsilon_0 \ell}{\ln(b/a)} \)

- What if the gap is very narrow i.e., \( d = b - a \) is small
  \[
  \ln\left(\frac{b}{a}\right) = \ln\left(\frac{a+d}{a}\right) = \ln\left(1 + \frac{d}{a}\right)
  \]

- Taylor expansion tells us
  \[
  \ln\left(1 + \frac{d}{a}\right) \approx \ln(1) + \frac{d}{a} \ln(x) \bigg|_{x=1} \times \frac{d}{a} = \frac{d}{a}
  \]

\[
C \approx \frac{2\pi \varepsilon_0 \ell a}{d} = \frac{\varepsilon_0 A}{d}
\]

\( A = 2\pi a \ell = \text{Area of the inner cylinder} \)

Same as parallel-plate
Capacitor Arithmetic

- Electrical circuits get tedious to draw

  ![Capacitor schematic](image)

- Let’s consider combinations of capacitors

  - Parallel
  - Series
Parallel Combination

- Connect two capacitors $C_1$ and $C_2$ in parallel
  - Potential $\Delta V$ across $C_1$ and $C_2$ are the same
  - $C_1$ and $C_2$ hold charges
    \[ Q_1 = C_1 \Delta V \quad Q_2 = C_2 \Delta V \]
  - Combined, they looks like a capacitor that holds
    \[ Q = Q_1 + Q_2 = (C_1 + C_2) \Delta V \]
- Parallel Capacitor Rule:
  \[ C_{\text{equiv.}} = C_1 + C_2 \]
Series Combination

Connect two capacitors $C_1$ and $C_2$ in series

- Charge $Q$ in $C_1$ and $C_2$ are the same
  - Can you tell why?

- $C_1$ and $C_2$ hold charges
  \[ Q = C_1 \Delta V_1 \quad Q = C_2 \Delta V_2 \]

- Total potential $\Delta V$ must be sum of $\Delta V_1$ and $\Delta V_2$
  \[ \Delta V = \Delta V_1 + \Delta V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} \]

- Series Capacitor Rule:
  \[ \frac{1}{C_{\text{equiv.}}} = \frac{1}{C_1} + \frac{1}{C_2} \]
For arbitrary number of capacitors

\[ C_{\text{equiv.}} = C_1 + C_2 + C_3 + \cdots + C_n \]

\[ \frac{1}{C_{\text{equiv.}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n} \]

**Ex:**

\[ \frac{1}{\frac{1}{2 \mu F} + \frac{1}{5 \mu F} + \frac{1}{3 \mu F}} = \frac{1}{2 \mu F} + \frac{1}{5 \mu F} = \frac{1}{2.5 \mu F} \]
Energy in a Capacitor

- Capacitor \( C \) is holding charge \( q \)
- We move small charge \( dq \) from the negative plate to the positive plate
- Moving charge across potential requires work:

\[
 dW = dq \Delta V = dq \frac{q}{C}
\]

- Question: starting from \( q = 0 \), how much work is needed to charge up the capacitor until \( q = Q \)?
- Answer:

\[
 W = \int_{0}^{Q} \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{1}{2} C (\Delta V)^2
\]

Capacitor must be storing this energy
Capacitor as a Storage Device

- Capacitor can hold energy \( U = \frac{1}{2} C(\Delta V)^2 \)
  - Not a lot for typical capacitance and voltage found in electric circuits
  - It can get big with high-voltage circuits
    - Don’t open up your TV set!
  - It may be sufficient for low-current circuits
    - “Battery-less” LED flashlights that light up by shaking
- Capacitors can be used for storing “information”
  - Charge up \(\rightarrow\) Check the voltage later
    - Computer memory chips (dynamic RAMs) contain arrays of microscopic capacitors
Where is the Energy

- When a capacitor is charged up, there is energy $U$ in it
  - Exactly where?
  - A capacitor is empty inside
- All what’s inside is $E$ field
  - Let’s assume the energy is uniformly distributed between the plates

Energy density $u = \frac{U}{Ad} = \frac{C(\Delta V)^2}{2Ad} = \frac{C(Ed)^2}{2Ad}$

$$u = \frac{1}{2} \varepsilon_0 E^2$$

Empty space with electric field $E$ is filled with energy density $\frac{1}{2} \varepsilon_0 E^2$
Energy in the Field

- Empty space holding energy may sound strange
- Consider potential energy due to gravity

\[ \text{This object has a potential energy } mgh \]

But how does it store the energy? The object itself does not change its properties!

- Potential energy is not in the object, but in its relationship with the gravitational field around it
- Gravitational field \( g \) (like \( E \) field) has energy density that is proportional to \( g^2 \)
- It’s just another way of looking at forces
Summary

- Defined capacitance $Q = C\Delta V$
  - For parallel-plate capacitor $C = \frac{\varepsilon_0 A}{d}$

- Capacitor arithmetic
  - Parallel or serial $\frac{1}{C_{\text{equiv.}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}$

- Energy stored in a charged capacitor
  - It’s in fact in the gap, where the energy density is $u = \frac{1}{2} \varepsilon_0 E^2$
  - $U = \frac{1}{2} C(\Delta V)^2$