Physics 11b
Lecture #17

Electromagnetic Radiation

S&J Chapter 34
What We Did Last Time

- Impedance
  \[ Z_R = \frac{\Delta V_R}{I_R} = R \]
  \[ Z_C = \frac{\Delta V_C}{I_C} = \frac{1}{\omega C} \]
  \[ Z_L = \frac{\Delta V_L}{I_L} = \omega L \]

- C and L are frequency dependent

- RLC circuit
  \[ Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \]

- Resonance at
  \[ \omega_0 = \frac{1}{\sqrt{LC}} \]
  , with quality factor
  \[ Q = \frac{\omega_0}{\Delta \omega} = \frac{\omega_0 L}{R} \]

- Transformer
  \[ \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1} \]
  \[ \frac{I_1}{I_2} = \frac{N_2}{N_1} \]
  \[ R_{\text{equiv.}} = \left(\frac{N_1}{N_2}\right)^2 \frac{R}{R} \]
Today’s Goals

- Maxwell’s Equations
  - Complete the “fundamental E&M equations”
  - Predict electromagnetic radiation
    - That’s light, radio waves, etc.
- Electromagnetic waves
  - Study the plane-wave solution
  - Energy and momentum carried by EM waves
Four E&M Equations

- We’ve summarized them in Lecture #13

- Gauss’s law for $\mathbf{E}$
  $$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}$$
  Surface integral is taken over a closed surface $S$

- Gauss’s law for $\mathbf{B}$
  $$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$
  Line integral is taken around a closed loop

- Faraday’s law
  $$\oint \mathbf{E} \cdot ds = -\frac{d\Phi_B}{dt}$$

- Ampère’s law
  $$\oint \mathbf{B} \cdot ds = \mu_0 I$$

- Maxwell (1831-1879) noticed something was wrong
Electricity and magnetism appear very similar except for one thing: there is no magnetic charge. This explains some differences between the equations:

\[ \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0} \quad \text{but} \quad \oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \]

\[ \oint_c \mathbf{B} \cdot ds = \mu_0 I \quad \text{but} \quad \oint_c \mathbf{E} \cdot ds = -\frac{d\Phi_B}{dt} \]

What happened to this one?

The symmetry would be perfect if it weren’t for the

Maxwell: “I’ll add an extra term just so that the equations become symmetric!”
Maxwell’s Equations

- The extra term is called the displacement current
  - Name is historic – Don’t worry about it
  - Maxwell’s reasoning is explained in textbook 30.7
- The extra term opens a new way of producing $B$ field by time-variation of the $E$ field
  - We already know induction: varying $B$ field creates $E$ field
  - Can $E$ field and $B$ field exist without charge or current?
Consider a simple case: Empty space

\[ \oint_{S} \mathbf{E} \cdot d\mathbf{A} = 0 \quad \oint_{S} \mathbf{B} \cdot d\mathbf{A} = 0 \]

\[ \oint_{S} \mathbf{E} \cdot ds = -\frac{d\Phi_{B}}{dt} \quad \oint_{S} \mathbf{B} \cdot ds = \mu_{0}\varepsilon_{0} \frac{d\Phi_{E}}{dt} \]

Let’s assume \( \mathbf{E} \) and \( \mathbf{B} \) depends on \( x \) and \( t \)

- But not \( y \) and \( z \)

Draw a small loop in \( x-y \) plane and calculate the integrals

- Assume \( \Delta x \) and \( \Delta y \) are very small
E&M Fields in Free Space

- **Faraday’s law**

\[ \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_{B}}{dt} \]

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \int_{a}^{b} E_{x} \, dx + \int_{b}^{c} E_{y} \, dy + \int_{c}^{d} E_{x} \, dx + \int_{d}^{a} E_{y} \, dy \]

\[ = \left( E_{y}(x + \Delta x) - E_{y}(x) \right) \Delta y \]

\[ \approx \frac{\partial E_{y}}{\partial x} \Delta x \Delta y \]

Remember

\[ \frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \]

\[ \Phi_{B} = B_{z} \Delta x \Delta y \]

\[ \frac{\partial E_{y}}{\partial x} = -\frac{\partial B_{z}}{\partial t} \]
Ampere’s law \( \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \) works the same way

- Just the sign plus the constant
- From the loop in the \( x-y \) plane \( \Rightarrow \)
- Use a loop in the \( x-z \) plane

\[
\frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\]

Minus sign comes from the fact that \( \hat{x} \times \hat{y} = \hat{z} \) but \( \hat{x} \times \hat{z} = -\hat{y} \)

We have a pair of equations

\[
\begin{align*}
\frac{\partial E_y}{\partial x} &= -\frac{\partial B_z}{\partial t} \\
\frac{\partial B_z}{\partial x} &= -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\end{align*}
\]
E&M Fields in Free Space

Eliminate $B_z \Rightarrow$ $

\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$

Eliminate $E_y \Rightarrow$ $

\frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$

These are general wave equations, for which the simplest solutions are sinusoidal waves

Solutions are $

E_y = E_{\max} \cos(kx - \omega t)$

$B_z = B_{\max} \cos(kx - \omega t)$

Plug into the wave equations

$\frac{\partial^2 \cos(kx - \omega t)}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \cos(kx - \omega t)}{\partial t^2}$

$k^2 = \mu_0 \varepsilon_0 \omega^2$
How does $\cos(kx - \omega t)$ looks like?

- Sinusoidal waves at any moment
- Wave length $\lambda$ is given by
  
  \[ k\lambda = 2\pi \quad \Rightarrow \quad \lambda = \frac{2\pi}{k} \]

- $k$ is the wave number
- The waves move toward $+x$ direction with time
  
  - $\omega$ is the angular frequency as usual
Speed of the Waves

- We can calculate the speed of the waves
  \[ v = \frac{\lambda}{T} \quad \lambda = \frac{2\pi}{k} \quad T = \frac{2\pi}{\omega} \]
  \[ v = \frac{\omega}{k} \]

- We’ve found
  \[ k^2 = \mu_0 \varepsilon_0 \omega^2 \]
  \[ v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]
  \[ = 2.998 \times 10^8 \text{ m/s} \]

- Maxwell’s discovery: Electromagnetic fields can exist in free space as waves propagating with the speed of light
  - Light *must* be electromagnetic waves
Closer Look on the Waves

\[ E_y = E_{\text{max}} \cos(kx - \omega t) \quad \quad B_z = B_{\text{max}} \cos(kx - \omega t) \]

Plug into

\[ \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \quad \quad kE_{\text{max}} = \omega B_{\text{max}} \]

At any point on the waves, \( E \) and \( B \) are always proportional to each other, and the ratio is \( c \)

\[ \frac{E_{\text{max}}}{B_{\text{max}}} = \frac{\omega}{k} = c \]
The EM waves discussed so far are plane waves:

\[ E_y = E_{\text{max}} \cos(kx - \omega t) \]
\[ B_z = B_{\text{max}} \cos(kx - \omega t) \]

- It depends on \( x \), but uniform in the \( y-z \) plane
- It moves along \(+x\)
- Note that \( E \), \( B \), and \(+x\) make a right-hand
  - i.e., \( E \times B \) points \(+x\)
  - Obviously there is another independent solution:
\[ E_z = E_{\text{max}} \cos(kx - \omega t) \]
\[ B_y = -B_{\text{max}} \cos(kx - \omega t) \]
Energy in the Waves

- E and B fields hold energies
  - Energy densities are given by
  - In the EM waves, $E = cB$

$$u_E = \frac{\varepsilon_0 (cB)^2}{2} = \frac{B^2}{2\mu_0} = u_B$$

Remember $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

- Equal energies in E and B
- Total energy density is $u = u_E + u_B = \varepsilon_0 E^2 = \frac{B^2}{\mu_0}$

- E and B vary with position and time
- Average energy density is $u_{ave} = \frac{\varepsilon_0 E_{max}^2}{2} = \frac{B_{max}^2}{2\mu_0}$
Energy Carried by the Waves

- EM waves hits an area $A \rightarrow$ How much energy?
  - Each second, energy in the box of $A \times c$ arrives
    \[
    \text{Power } P = u_{\text{ave}} A c
    \]
- Define **intensity** as the energy arriving in a unit area per second
  \[
  I = \frac{P}{A} = u_{\text{ave}} c = \frac{c B_{\text{max}}^2}{2 \mu_0} = \frac{E_{\text{max}} B_{\text{max}}}{2 \mu_0}
  \]
  - Unit: W/m$^2$
  - Example: direct sunlight on earth is about 1kW/m$^2$
    - What’s $E_{\text{max}}$ and $B_{\text{max}}$?
In EM waves, \( \mathbf{E} \) and \( \mathbf{B} \) fields are perpendicular to the direction of propagation, which is given by \( \mathbf{E} \times \mathbf{B} \).

- Define Poynting vector: \( \mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \).
- It points the direction of propagation.
- Magnitude is the intensity = energy flow per area per second.
- Time-average is 

\[
S_{\text{ave}} = \frac{E_{\text{max}} \times B_{\text{max}}}{2\mu_0}
\]
Momentum and Pressure

- EM waves carry momentum as well as energy
  - \[ p = \frac{E}{c} \]
  - Object absorbing light is pushed by the light pressure
  - Object emitting light is pushed back
- If power of the light is \( P \), force \( F \) is
  - \[ F = \frac{P}{c} \]
- e.g. a 100 MW laser pulse produces a force of
  - \[ 1 \times 10^8 \text{ (W)} / 3 \times 10^8 \text{ (m/s)} = 0.3 \text{ (N)} \]
Spherical Waves

- Real-world EM waves are (often) not plane waves
  - Light from small sources, radio waves from an antenna, etc.
  - They usually spread spherically
- **Spherical waves** behave pretty much the same as plane waves, except:
  - It gets weaker as it goes

\[
E \propto \frac{1}{r} \quad B \propto \frac{1}{r} \quad I \propto \frac{1}{r^2}
\]

- All what we discussed about plane waves applies to spherical waves

This is obvious because of energy conservation
Summary

- Maxwell’s Eqns.

\[ \oint E \cdot dA = \frac{q}{\varepsilon_0} \quad \oint B \cdot dA = 0 \]

\[ \oint E \cdot ds = -\frac{d\Phi_B}{dt} \quad \oint B \cdot ds = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]

- In free space, predict electromagnetic waves

\[ E_y = E_{\text{max}} \cos(kx - \omega t) \]
\[ B_z = B_{\text{max}} \cos(kx - \omega t) \]

- Speed

\[ v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \]

- Poynting vector

\[ S = \frac{E \times B}{\mu_0} \]
gives the energy flow

- Energy and momentum related by

\[ \text{momentum} = \frac{\text{energy}}{c} \]