Mechanics
Physics 151

Lecture 2
Elementary Principles
(Goldstein Chapter 1)

Administrivia

- First Problem Set
  - 3 problems for the section
  - Work on them before coming to your section!
  - 3 for the report (due next week)
- If you haven’t filled the Survey, please do it
  - We need it for sectioning and study-grouping
- Section time: Tue. 6 PM, 7 PM and Wed. 5 PM
  - If none of these slots works for you, let me know
  - Sectioning will be announced on Monday by email
  - We will also assign you into study groups (~6 each)

What We Did Last Time

- Reviewed basic principles of Newtonian Mechanics
  - Define standard notations and usages
  - Momenta, conservation laws, kinetic & potential energies
  - Concentrated on the motion of a single particle
Goals for Today

- Single \(\rightarrow\) multi-particle system
  - Force between particles
    - Laws of action and reaction (Newton's 3rd Law)
- Introduce constraints
  - Holonomic and nonholonomic constraints
- Introduce Lagrange's Equation

System of Particles

- More than one particles? \(\rightarrow\) Just add indices!
  \[ \mathbf{F} = \mathbf{p} \quad \mathbf{N} = \mathbf{L} \]
- Subtlety: \(\mathbf{F}\) may be working between particles
  - Distinguish between internal and external forces

\[ \mathbf{F} = \sum_{j} \mathbf{F}_j + \mathbf{F}^{\text{ext}} \]

- Force acting on particle \(i\)
- Force from particle \(j\)
- Force from outside
- Now add up over \(i\) to see the overall picture

Sum of Particles

\[ \sum_{i,j} \mathbf{F}_{ij} + \sum_{i} \mathbf{F}^{\text{int}} = \sum_{i,j} (\mathbf{F}_j + \mathbf{F}_i) + \sum_{i} \mathbf{F}^{\text{ext}} \]

- This term vanishes if \(\mathbf{F}_j = -\mathbf{F}_i\)
  - Weak law of action and reaction

Forces two particle exert on each other are equal and opposite

\[ \sum_{i} \mathbf{F}_i = \sum_{i} \mathbf{F}^{\text{int}} \]

C.f. the strong law of action and reaction

Forces two particle exert on each other are equal, opposite, and along the line joining the particles
Sum of Particles

- Now consider the equations of motion
  \[
  \sum F = \sum F^{(i)} = \sum \dot{p}_i = \sum \frac{d}{dt} \sum m_i r_i
  \]

- Define center of mass
  \[
  R = \sum \frac{m_i r_i}{m} = \sum \frac{m_r_i}{M}
  \]

\[
\text{MR} = \sum \dot{p}^{(i)} = \dot{F}^{(i)}
\]

Center of mass moves like a particle of mass \(M\) under total external force \(F^{(i)}\)

Total Linear Momentum

- The sum of the linear momenta is
  \[
  \mathbf{P} = \sum \dot{p}_i = \sum \frac{d}{dt} \sum m_i \mathbf{r}_i = \mathbf{MR}
  \]

- Taking the time derivative
  \[
  \dot{\mathbf{P}} = \frac{d}{dt} \mathbf{MR} = \dot{\mathbf{F}}^{(i)}
  \]

- Conservation of total linear momentum

  If the total external force \(F^{(i)}\) is zero, the total linear momentum \(\mathbf{P}\) is conserved.

  Assumed weak law of action & reaction

Total Angular Momentum

- The sum of the angular momenta is
  \[
  \mathbf{L} = \sum \mathbf{L}_i = \sum \mathbf{r}_i \times \dot{p}_i
  \]

- Take time derivative and use
  \[
  \dot{\mathbf{p}}_i = \mathbf{F}_i = \sum \mathbf{F}_j + \mathbf{F}^{(i)}
  \]

\[
\mathbf{L} = \sum \mathbf{r}_i \times \mathbf{F}_j + \sum \mathbf{r}_i \times \mathbf{F}^{(i)}
\]

- This term vanishes only if \(\mathbf{F}_j\) satisfies the strong law of action and reaction

Total external torque
Assuming strong law of action and reaction

\[ L = \sum \mathbf{r} \times \mathbf{F} = \sum N^{\alpha} = N^{\alpha} \]

→ Conservation of total angular momentum

If the total external torque \( N^{\alpha} \) is zero, the total angular momentum \( L \) is conserved.

A multi-particle system (= extended object) can be treated as if it were a single particle if the internal forces obey the strong law of action and reaction.

Most forces we know obey strong law of action and reaction

- Gravity, electrostatic force

There are rare exceptions

- E.g. Lorenz force felt by moving charges
- Conservation of linear & angular momenta fails

Take into account the EM field

- Particles exchange forces with the field
- The field itself has linear & angular momenta

→ Conservation laws restored

We will see (in 2 lectures) that \( P \) and \( L \) must be conserved if the laws of physics are isotropic in space

- No special origin
- No special orientation

If we accept these symmetries as fundamental principles, all forces must satisfy the action-reaction laws → “Proof” of Newton’s 3rd Law.
Total Angular Momentum

- Define particle \( i \)'s position from the center of mass
  \[ r'_i = r_i - R \]
- Also define the velocities \( v'_i = v_i \) \( v = R \)
- Calculate the total angular momentum
  \[ L = \sum r'_i \times p_i = \sum (r_i + R) \times m (v_i + v) \]
  \[ L = R \times M v + \sum r'_i \times m v'_i \]

Angular momentum of motion concentrated at the center of mass
Angular momentum of motion around the center of mass

Kinetic Energy

- The work done by force \( W_{ij} = \sum \int F \cdot ds \)
- Positions 1 and 2 are now configurations (sets of positions)
- Use equations of motion to derive
  \[ W_{ij} = T_j - T_i \] where \( T = \sum \frac{1}{2} m v_i^2 \)
- One can split \( T \) into two pieces
  \[ T = \sum \frac{1}{2} m (v + v') \cdot (v + v') = \frac{1}{2} M v^2 + \sum \frac{1}{2} m v_i^2 \]

Motion concentrated at the center of mass
Motion around the center of mass

Potential Energy

- Assume conservative external force \( F^{ext} = -\nabla V \)
  \[ \sum_i \int F^{ext} \cdot ds = -\sum_i \int \nabla V_i \cdot ds = -\sum_i V_i \]
- Assume also conservative internal forces \( F_{ij} = -\nabla V \)
  - To satisfy strong law of action/reaction
  \[ V_i = V_i(|r_i - r_j|) \] Potential depends only on the distance

\[ \sum_{i,j} \int F_{ij} \cdot ds = -\sum_{i,j} \int \nabla V_i \cdot ds = -\frac{1}{2} \sum_i \nabla V_i \] Bit of work
\[ -\frac{1}{2} \sum \nabla V_i^2 \]
Energy Conservation

- If all forces are conservative, one can define total potential energy:
  \[ V = \sum_{i} V_i + \frac{1}{2} \sum_{i \neq j} V_{ij} \]
  - Then the total energy \( T + V \) is conserved
  - The second term is internal potential energy
    - It depends on the distances between all pairs of particles
    - Constant if particles' relative configuration is fixed
      \[ \text{Rigid bodies} \]

Constraints

- Equation of motion \( m \ddot{r}_i = F_i = F_i^{(c)} + \sum_{j} F_{ij} \) assumes that particles can move anywhere in space
  - Not generally true
    - In fact never true – Free space is an idealization
    - Amusement-park ride constrained (hopefully) on a rail
    - Billiard balls on a pool table
  - How can we accommodate constraints in the equation of motion?
    - Depends on the type of the constraint

Holonomic Constraints

- Constraints may be expressed by
  \[ f(\mathbf{r}_i, \mathbf{r}_j, \ldots, t) = 0 \]
  - A holonomic constraint
    - Particle on the x-y plane \( z = 0 \)
    - Rigid body \( (\mathbf{r}_i - \mathbf{r}_j)^T \mathbf{c}_i = 0 \)
  - All other cases are called nonholonomic
    - It means “we don’t really want to mess with it”
    - May be inequalities such as \( z \geq 0 \)
    - May depend on derivatives such as \( \dot{r}_i \)
  - We will deal only with holonomic constraints
Independent Variables

- A holonomic constraint reduces the number of independent variables by 1
- If \( z = 0 \), you are left with only \( x \) and \( y \)
- You may be able to solve the constraint for one variable
  \[ f(x, y, z, r_1, r_2, \ldots) = 0 \quad \Rightarrow \quad y = g(x, z, r_1, r_2, \ldots) \]
  - Then you can drop this variable
- You may have to switch to a different set of variables
- For a particle on a sphere \( x^2 + y^2 + z^2 = c^2 \) a good choice is \((\theta, \phi)\)
- New set of variables \( \Rightarrow \) Generalized coordinates

Generalized Coordinates

- \( N \) particles have \( 3N \) degrees of freedom
- Introducing \( k \) holonomic constraints reduces it to \( 3N - k \)
- Using generalized coordinates \( q_1, q_2, \ldots, q_{3N-k} \)

\[
\mathbf{r} = \mathbf{r}(q_1, q_2, \ldots, t)
\]

Transformation equations from \((r_i)\) to \((q_j)\)

- Example:
  \[
  \begin{align*}
  x &= c \sin \theta \cos \phi \\
  y &= c \sin \theta \sin \phi \\
  z &= c \cos \theta 
  \end{align*}
  \]

Transformation from \((x, y, z)\) to \((\theta, \phi)\)

Now What?

- We know the equations of motion for \((r_i)\)
  \[
  m_\mathbf{a} = \mathbf{F} = \mathbf{F}_c + \sum \mathbf{F}_\ell
  \]
- We know how to include constraints by switching to generalized coordinates
  \[
  \mathbf{r} = \mathbf{r}(q_1, q_2, \ldots, t)
  \]
- How can we transform the equation of motion to the generalized coordinates?
  \[
  \text{Lagrange’s Equations}
  \]
Why Constraints?

- Constraint is an idealized classical concept
- Nothing is perfectly constrained in QM
- How useful is it to switch between coordinates?

Constraints and Force

- A holonomic constraint is an infinitely strong force
  - Or an infinitely high potential wall
- Reality is always smoother
  - E.g. electron of a hydrogen atom

<table>
<thead>
<tr>
<th>$V(r)$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>free</td>
<td>constrained</td>
</tr>
</tbody>
</table>

It's still true that the electron feels strong radial (binding) force, while it can move freely around the nucleus.

Force and Symmetry

- Without forces, all coordinate systems are equal
  - $x$-$y$-$z$ system is the simplest
- Forces break the symmetry
  - Some coordinate system works better than others
- Generalized coordinates offer natural way of handling systems with such forces
- Constraints are extreme cases
  - We develop our technique with them

OK, back to the business…
Lagrange’s Equations

Express \( L = T - V \) in terms of generalized coordinates \( \{q_i\} \), their time-derivatives \( \{\dot{q}_i\} \), and time \( t \)
- The potential \( V = V(q, t) \) must exist
- i.e. all forces must be conservative
- Let’s do a quick example to see how it works

Ex: Particle on a Line

A particle moving on the \( x \)-axis \( x = x(t), y = 0, z = 0 \)
- Kinetic and potential energies:
  \[ T = \frac{m}{2} \dot{x}^2 \quad V = V(x) \]
  \[ L = \frac{m}{2} \dot{x}^2 - V(x) \]
- Equivalent to Newton’s Eqn given that \( F_x = -\frac{\partial V}{\partial x} \)

OK, it works

Summary

- Discussed multi-particle systems
- Internal and external forces
  - Laws of action and reaction
  - Momenta, conservation laws, kinetic & potential energies
- Introduced constraints
  - Holonomic and nonholonomic constraints
  - Generalized coordinates
- Introduced Lagrange’s Equations
  - Next: Prove that Lagrange’s and Newton’s Equations are equivalent