Mechanics
Physics 151

Lecture 4
Hamilton’s Principle
(Chapter 2)

Administrivia

- Problem Set #1 due
  - Solutions will be posted on the web after this lecture
- Problem Set #2 is here
  - Due next Thursday
- Next lecture (Tuesday) will be given by Srinivas and Abdol-Reza
  - I will be attending a workshop at Stanford
What We Did Last Time

- Derived **Lagrange’s Eqn** from Newton’s Eqn
  - Using **D’Alembert’s Principle** = differential approach
- **Lagrange’s Equations** work if
  - Constraints are **holonomic** → Generalized coordinates
  - Forces of constraints do no work → **No frictions**
  - Other forces are **monogenic** → Generalized potential

\[ Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \]

Today’s Goals

- Discuss **Hamilton’s Principle**
  - Derive Lagrange’s Eqn from Hamilton’s Principle
  - **Calculus of variation**
  - Looks unfamiliar, but not so difficult
- Discuss **conservation laws** again
  - Using Lagrangian formalism
  - Linear, angular momenta
  - Connection between **symmetry, invariance** of the Lagrangian, and **conservation** of generalized momentum
Configuration Space

- Generalized coordinates $q_1,...,q_n$ fully describe the system’s configuration at any moment.
- Imagine an $n$-dimensional space.
  - Each point in this space ($q_1,...,q_n$) corresponds to one configuration of the system.
  - Time evolution of the system $\rightarrow$ A curve in the configuration space.

Action Integral

- A system is moving as $q_j = q_j(t) \quad j = 1...n$.
- Lagrangian is $L(q, \dot{q}, t) = L(q(t), \dot{q}(t), t) = L(t)$.
- Action $I$ depends on the entire path from $t_1$ to $t_2$.
- Choice of coordinates $q_j$ does not matter.
  - Action is invariant under coordinate transformation.

$$I = \int_{t_1}^{t_2} L dt$$  Action, or action integral.
Hamilton’s Principle

The action integral of a physical system is stationary for the actual path

- This is equivalent to Lagrange’s Equations
  - We will prove this
- Three equivalent formulations
  - Newton’s Eqn depends explicitly on x-y-z coordinates
  - Lagrange’s Eqn is same for any generalized coordinates
  - Hamilton’s Principle refers to no coordinates
    - Everything is in the action integral

Hamilton’s Principle is more fundamental probably...

Stationary

- Consider two paths that are close to each other
  - Difference is infinitesimal
  - Stationary means that the difference of the action integrals is zero to the 1st order of $\delta q(t)$
    - Similar to “first derivative = 0”

$$\delta t = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$$

- Almost same as saying “minimum”
  - It could as well be maximum

configuration space

$q(t)$

$q(t) + \delta q(t)$

$t_1$

$t_2$

$\delta q(t_1) = \delta q(t_2) = 0$
Infinitesimal Path Difference

- What’s $\delta q(t)$?
  - It’s arbitrary … sort of
  - It has to be zero at $t_1$ and $t_2$
  - It’s well-behaving

- Have to shrink it to zero
  - Trick: write it as $\delta q(t) = \alpha \eta(t)$
    - $\alpha$ is a parameter, which we’ll make $\rightarrow 0$
    - $\eta(t)$ is an arbitrary well-behaving function

  $\eta(t_1) = \eta(t_2) = 0$

Hamilton $\rightarrow$ Lagrange

- Consider 1 generalized coordinate $q$
  - Add $\delta q(t)$ to $q(t)$, then make $\delta q(t) \rightarrow 0$
  - Do this by $\delta q(t) = \alpha \eta(t)$
    - $\alpha$ is a parameter $\rightarrow 0$
    - $\eta(t)$ is an arbitrary well-behaving function

  $\eta(t_1) = \eta(t_2) = 0$

- Let’s define $I(\alpha) = \int_{t_1}^{t_2} L(q(t, \alpha), q(t, \alpha), t) dt$

NB: this also depends on $\eta(t)$
Calculus of Variations

- Let’s define \( I(\alpha) = \int_{t_1}^{t_2} L(q(t, \alpha), \dot{q}(t, \alpha), t) \, dt \)
  
  \( \text{NB: this also depends on } \eta(t) \)

- If the action is stationary

  \[
  \left( \frac{dI(\alpha)}{d\alpha} \right)_{\alpha=0} = 0
  \]
  
  for any \( \eta(t) \)

  \[
  q(t, \alpha) = q(t) + \alpha \eta(t)
  \]

Some work!

\[
\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \frac{dq}{dt} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta(t) \, dt = 0
\]

Arbitrary function

Lagrange’s Equation

- Fundamental lemma

  \[
  \int_{x_1}^{x_2} M(x) \eta(x) \, dx = 0 \text{ for any } \eta(x)
  \]

  \( M(x) = 0 \text{ for } x_1 < x < x_2 \)

- We got

  \[
  \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \frac{dq}{dt} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta(t) \, dt = 0
  \]

  \[
  \frac{\partial L}{\partial q} \frac{dq}{dt} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0
  \]

Done!
Notation of Variation

- For shorthand, we use $\delta$ for infinitesimal variation
  - i.e. $\alpha$-derivative at $\alpha = 0$
    \[
    \delta I \equiv \left( \frac{dI}{d\alpha} \right)_{\alpha=0} \int_0^t L(q(t,\alpha),\dot{q}(t,\alpha),t) \, dt \, d\alpha
    \]
    \[
    \delta q \equiv \left( \frac{dq}{d\alpha} \right)_{\alpha=0} \, d\alpha = \eta(t) \, d\alpha
    \]
- Hamilton’s Principle can be written as
  \[
  \delta I = \int_0^t \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt = 0
  \]

Going Multi-Co-Ordinates

- Trivial to expand $q \rightarrow (q_1, q_2, \ldots, q_n)$
  - See Goldstein Section 2.3
    \[
    \delta I = \int_0^t \sum_i \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt = 0
    \]
    = 0 for each $i$
- Assumption: $\delta q_1, \delta q_2, \ldots$ are arbitrary and independent
  - Not true for $x$-$y$-$z$ coordinates if there are constraints
  - True for generalized coordinates if the system is holonomic
Hamilton’s Principle

\[ \delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0 \]

- Action \( I \) describes the entire motion of the system
  - It is sufficient to derive the equations of motion
- Action \( I \) does not depend on the choice of the coordinates
  - Lagrange formalism is coordinate invariant
- Adding \( dF/dt \) to \( L \) would add \( F(t_2) - F(t_1) \) to \( I \)
  - It wouldn’t affect \( \delta I \) ← Variations are 0 at \( t_1 \) and \( t_2 \)
- Arbitrarity of \( L \) is obvious

Calculus of Variation

- Technique has wider applications
  - In general for \( J = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx \)
    \[ y' = \frac{dy}{dx} \]
    \[ \delta J = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y} \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0 \]
  - Examples in Goldstein Section 2.2
  - Most famous: the brachistochrone problem

Fastest path via gravity
Conservation Laws

- We’ve seen (in Lectures 1&2) conservation of linear, angular momenta and energy in Newtonian mechanics
  - How do they work with Lagrange’s equations?
  - Should better be the same…
- We’ll find a few differences and assumptions
  - They are, in fact, limitations we ignored so far

Momentum Conservation

- Let’s consider a simple system

\[ L = T - V = \sum_i m \left( \frac{\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2}{2} \right) - V(x_i, y_i, z_i, t) \]

- Potential does not depend on velocity
- Momentum \( p_{ix} \) conserved if \( V \) does not depend on \( x_i \)
- Now try to generalize from here
Generalized Momentum

- Let’s call \( p_j = \frac{\partial L}{\partial \dot{q}_j} \) the generalized momentum
  - Also known as canonical or conjugate momentum
  - Equals to usual momentum for simple x-y-z coordinates
- Lagrange’s equation becomes \( \frac{dp_j}{dt} - \frac{\partial L}{\partial q_j} = 0 \)
  - \( p_j \) is conserved if \( L \) does not depend explicitly on \( q_j \)
  - Such \( q_j \) is called cyclic (or ignorable)

Generalized momentum associated with a cyclic coordinate is conserved

Linear momentum conservation is a special case

Generalized Momentum

- Generalized momentum may not look like linear momentum
  - Dimension may vary, if \( q_j \) is not a space coordinate
    - \( p_jq_j \) always has the dimension of action (= work × time)
  - Form may vary if \( V \) depends on velocity
    - Example: a particle in EM field
      \[
      L = \frac{1}{2}mv^2 - q\phi + qA \cdot v
      \]
      \[
      p_x = m\dot{x} + qA_x
      \]
      Extra term due to velocity-dependent potential
Symmetry

- Linear momentum $\mathbf{p} = (p_x, p_y, p_z)$ is conjugate of $(x, y, z)$ coordinates
  - Conserved if Lagrangian does not depend explicitly on position
  - I.e. if Lagrangian is invariant under space translation $(x, y, z) \rightarrow (x + \Delta x, y + \Delta y, z + \Delta z)$
  - Such a system is called symmetric under space translation
- Symmetry of a system $\rightarrow$ Invariance of Lagrangian $\rightarrow$ Conservation of conjugate momentum
  - Let’s study an example of angular momentum

Angular Momentum

- Consider a multi-particle system $\mathbf{r}_i = r_i(q_1, ..., q_n, t)$
  - Suppose $q_1$ turns the whole system around
  - Example: $\phi$ in $\mathbf{r}_i = (x_i, y_i, z_i) = (r_i \cos \phi, r_i \sin \phi, z_i)$
  - Assume $V$ does not depend on $\phi$
- Conjugate momentum is
  \[
  p_{\phi} \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}}
  \]
  \[
  = \mathbf{n} \cdot \sum_i \mathbf{L}_i = \mathbf{n} \cdot \mathbf{L}
  \]
  Axis of rotation
  Bit of work
  Total angular momentum
  $\mathbf{r}_i(\phi + d\phi)$
  $\mathbf{r}_i(\phi)$
Angular Momentum

- Angular momentum is conserved if the system is symmetric under rotation
  - How does this relate to the total torque $N$?

  - $T$ cannot depend on $\phi$ $\iff$ Rotating doesn’t change $v_i^2$

  $\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi} = \sum_i F_i \cdot \frac{\partial r_i}{\partial \phi} = \sum_i F_i \cdot (n \times r_i) = n \cdot \sum_i r_i \times F_i$

  Total torque is zero along the axis of symmetry
Conservation Laws

- Following statements are equivalent:
  - System is symmetric wrt a generalized coordinate
  - The coordinate is cyclic (does not appear in Lagrangian)
  - The conjugate generalized momentum is conserved
  - The associated generalized force is zero

<table>
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<tr>
<th>Symmetry</th>
<th>Spatial translation</th>
<th>Rotation</th>
</tr>
</thead>
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<td>Coordinate</td>
<td>Distance along an axis</td>
<td>Angle around an axis</td>
</tr>
<tr>
<td>Momentum</td>
<td>Linear</td>
<td>Angular</td>
</tr>
<tr>
<td>Force</td>
<td>Force</td>
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</tr>
</tbody>
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Summary

- Derived Lagrange’s Eqn from Hamilton’s Principle
  - Calculus of variation
- Discussed conservation laws
  - Generalized (conjugate) momentum \[ p_j = \frac{\partial L}{\partial \dot{q}_j} \]
  - Symmetry of the system
    - Invariance of the Lagrangian
    - Conservation of momentum
- We are almost done with the basic concepts
  - Finish up next Tuesday with energy conservation
  - Some applications are in order \( \rightarrow \) Central force problem