Lecture 5
Central Force Problem
(Chapter 3)
What We Did Last Time

- Introduced Hamilton’s Principle
  - Action integral is stationary for the actual path
  - Derived Lagrange’s Equations
    - Used calculus of variation
- Discussed conservation laws
  - Generalized (conjugate) momentum
  - Symmetry – Invariance – Momentum conservation
- We are almost done with the basic concepts
  - One more thing to cover …
Goals for Today

- Energy conservation
  - Define energy function
    - Subtle difference from the Newtonian version
- Central force problem ↦ First application
  - Motion of a particle under a central force
  - Simplify the problem using angular momentum conservation
- Discuss qualitative behavior of the solution
  - Use energy conservation
  - Distinguish bounded/unbounded orbits
- Actual solution ➔ Thursday
Consider time derivative of Lagrangian

\[ \frac{dL(q, \dot{q}, t)}{dt} = \sum_j \frac{\partial L}{\partial q_j} \frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t} \]

Using Lagrange’s equation one can derive

\[ \frac{\partial L}{\partial q_j} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \]

Define this as energy function \( h(q, \dot{q}, t) \)

Conserved if Lagrangian does not depend explicitly on \( t \)
“Energy” Function?

Does energy function represent the total energy?
- Let’s try an easy example first

Single particle moving along $x$ axis

\[
L = \frac{m\dot{x}^2}{2} - V(x)
\]

\[
h = m\dot{x}^2 - L = \frac{m\dot{x}^2}{2} + V(x) = T + V
\]

How general is this?

\[
h(q, \dot{q}, t) \equiv \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L
\]
Energy Function

Suppose $L$ can be written as

$$L(q, \dot{q}, t) = L_0(q, t) + L_1(q, \dot{q}, t) + L_2(q, \dot{q}, t)$$

- True in most cases of interest
- Derivatives satisfy

$$\frac{\partial L_0}{\partial \dot{q}_j} = 0 \quad \sum_j \dot{q}_j \frac{\partial L_1}{\partial \dot{q}_j} = L_1 \quad \sum_j \dot{q}_j \frac{\partial L_2}{\partial \dot{q}_j} = 2L_2$$

$$h(q, \dot{q}, t) \equiv \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = L_2 - L_0$$

Euler’s theorem
Energy Function

\[ h(q, \dot{q}, t) = L_2 - L_0 \]

\[ L = T - V \]

- Energy function equals to the total energy \( T + V \) if
  
  \[ T = L_2 \quad \text{and} \quad V = -L_0 \]

  - 1\textsuperscript{st} condition is satisfied if transformation from \( r_i \) to \( q_j \) is time-independent
  
  - 2\textsuperscript{nd} condition holds if the potential is velocity-independent
    
    \[ \text{No frictions} \rightarrow \text{Friction would dissipate energy} \]

- Let’s look into the 1\textsuperscript{st} condition
Kinetic Energy

\[ T = \sum_i \frac{m_i}{2} \dot{r}_i^2 \]

\[ r_i = r_i(q_1, \ldots, q_n) \]

- Using the chain rule
  \[ \frac{d\mathbf{r}_i}{dt} = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j \]

\[ \sum_i \frac{m_i}{2} \dot{r}_i^2 = \sum_i \frac{m_i}{2} \sum_{j,k} \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k = \sum_{j,k} \dot{q}_j \dot{q}_k \sum_i \frac{m_i}{2} \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \]

- This wouldn’t work if \[ r_i = r_i(q_1, \ldots, q_n, t) \] because
  \[ \frac{d\mathbf{r}_i}{dt} = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \]
Energy Conservation

- Energy function equals to the total energy if
  - Constraints are time-independent
    - Kinetic energy $T$ is $2^{nd}$ order homogeneous function of the velocities
  - Potential $V$ is velocity-independent
- Energy function is conserved if
  - Lagrangian does not depend explicitly on time
- These are restatement of the energy conservation theorem in a more general framework
  - Conditions are clearly defined

$h(q, \dot{q}, t) \equiv \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$
Consider a particle under a central force

- Force $F$ parallel to $r$

Assume $F$ is conservative $F = -\nabla V(r)$

- $V$ is function of $|r|$ if $F$ is central

Such systems are quite common

- Planet around the Sun
- Satellite around the Earth
- Electron around a nucleus

These examples assume the body at the center is heavy and does not move
Two-Body Problem

- Consider two particles without external force
  - \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) relative to center of mass
- Lagrangian is

\[
L = \frac{(m_1 + m_2)\dot{\mathbf{R}}^2}{2} + \sum_{i=1}^{2} \frac{m_i \dot{r}_i^2}{2} - V(r)
\]

Motion of CoM

Potential is function of
\[ |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1| \]

Strong law of action and reaction

\[
\mathbf{r}_1 = -\frac{m_2}{(m_1 + m_2)} \mathbf{r}
\]
\[
\mathbf{r}_2 = \frac{m_1}{(m_1 + m_2)} \mathbf{r}
\]

\[
\sum_{i=1}^{2} \frac{m_i \dot{r}_i^2}{2} = \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} \dot{\mathbf{r}}^2
\]
Two-Body $\rightarrow$ Central Force

\[ L = \frac{(m_1 + m_2)\dot{R}^2}{2} + \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} \dot{r}^2 - V(r) \]

- **R** is cyclic
  - CoM moves at a constant velocity
  - Move \( O \) to CoM and forget about it

Relative motion of two particles is identical to the motion of one particle in a central-force potential

- Reduced mass \( \mu = \frac{m_1 m_2}{(m_1 + m_2)} \) or \( \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \)
Hydrogen and Positronium

- Positronium is a bound state of a positron and an electron
  - Similar to hydrogen except $m(p) >> m(e^+)$
  - Potential $V(r)$ is identical
  - Turn them into central force problem

\[
\mu_{\text{positronium}} = \frac{m_p m_e}{(m_p + m_e)} = \frac{m_e}{2}
\]

\[
\mu_{\text{hydrogen}} = \frac{m_p m_e}{(m_p + m_e)} \approx m_e
\]

- Spectrum of positronium identical to hydrogen with $m_e \rightarrow m_e/2$

\[
V(r) = -\frac{q^2}{r}
\]
Spherical Symmetry

- Central-force system is **spherically symmetric**
  - It can be rotated around any axis through the origin
    - Lagrangian \( L = T(\dot{r}^2) - V(r) \) doesn’t depend on the direction
- Angular momentum is conserved \( \mathbf{L} = \mathbf{r} \times \mathbf{p} = \text{const} \)
  - Direction of \( \mathbf{L} \) is fixed
    - \( \mathbf{r} \perp \mathbf{L} \) by definition \( \rightarrow \) \( \mathbf{r} \) is always in a plane
- Choose polar coordinates
  - Polar axis = direction of \( \mathbf{L} \)
    - \( \mathbf{r} = \mathbf{r}(r, \theta, \psi) = \mathbf{r}(r, \theta) \)

Azimuth
Zenith = 1/2\( \pi \)
More Formally

- **Lagrangian in polar coordinates** \( \mathbf{r} = \mathbf{r}(r, \theta, \psi) \)

\[
L = T - V = \frac{m}{2} (r^2 + r^2 \sin^2 \psi \dot{\theta}^2 + r^2 \dot{\psi}^2) - V(r)
\]

- \( \theta \) is cyclic, but \( \psi \) is not

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = mr^2 (\ddot{\psi} - \sin \psi \cos \psi \dot{\theta}^2) = 0
\]

- We can choose the polar axis so that the initial condition is

\[
\psi = \frac{\pi}{2}, \dot{\psi} = 0 \quad \text{2nd term vanishes} \quad \ddot{\psi} = 0
\]

- Now \( \psi \) is constant. We can forget about it
Angular Momentum

\[ L = T - V = \frac{m}{2} \left( r^2 + r^2 \dot{\theta}^2 \right) - V(r) \]

- \( \theta \) is cyclic. Conjugate momentum \( p_\theta \) conserves
  \[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = \text{const} \equiv l \]

- Alternatively
  \[ \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \text{const} \]

- Kepler’s 2\textsuperscript{nd} law
- True for any central force
Radial Motion

\[
L = T - V = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)
\]

- Lagrange’s equation for \( r \) \( \rightarrow \)
  - Derivative of \( V \) is the force

\[
m\ddot{r} = mr\ddot{\theta}^2 + f(r)
\]

Centrifugal force

Central force

- Using the angular momentum \( l \)

\[
m\ddot{r} = \frac{l^2}{mr^3} + f(r)
\]

We know how to integrate this.
But we also know what we’ll get by integrating this

\[
\frac{d}{dt} (mr) - m r \dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0
\]

\[
f(r) = -\frac{\partial V(r)}{\partial r}
\]

\[
l = mr^2 \dot{\theta}
\]
Energy Conservation

\[ E = T + V = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) = \frac{m}{2} \dot{r}^2 + \frac{1}{2} \frac{l^2}{mr^2} + V(r) = \text{const} \]

\[ \dot{r} = \sqrt{\frac{2}{m} \left( E - V(r) - \frac{l^2}{2mr^2} \right)} \]

1st order differential equation of \( r \)

- One can solve this (in principle) by

\[ t = \int_0^t dt = \int_{r_0}^r \frac{dr}{\sqrt{\frac{2}{m} \left( E - V(r) - \frac{l^2}{2mr^2} \right)}} = t(r) \]

- Then invert \( t(r) \rightarrow r(t) \)

- Then calculate \( \theta(t) \) by integrating

\[ \dot{\theta} = \frac{l}{mr^2} \]

Done! (?)

NB: This never goes negative
A particle has **3 degrees of freedom**
- Eqn of motion is **2nd order** differential $\Rightarrow$ 6 constants

Each conservation law reduces one differentiation
- By saying “time-derivative equals zero”

We used $L$ and $E \Rightarrow$ 4 conserved quantities
- Left with **2 constants of integration** $= r_0$ and $\theta_0$

We don’t have to use conservation laws
- It’s just easier than solving all of Lagrange’s equations
Qualitative Behavior

- Integrating the radial motion isn’t always easy
  - More often impossible…

- You can still tell general behavior by looking at
  \[ V'(r) \equiv V(r) + \frac{l^2}{2mr^2} \]
  Quasi potential including the centrifugal force

- Energy \( E \) is conserved, and \( E - V' \) must be positive
  \[
  E = \frac{m\dot{r}^2}{2} + V'(r) \quad \Rightarrow \quad \frac{m\dot{r}^2}{2} = E - V'(r) > 0 \quad \Rightarrow \quad E > V'(r)
  \]

- Plot \( V'(r) \) and see how it intersects with \( E \)
Consider an attractive $1/r^2$ force

\[ f(r) = -\frac{k}{r^2} \quad \Rightarrow \quad V(r) = -\frac{k}{r} \]

- Gravity or electrostatic force

\[ V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2} \]

- $1/r^2$ force dominates at large $r$
- Centrifugal force dominates at small $r$
- A dip forms in the middle
Unbounded Motion

- Take $V'$ similar to $1/r^2$ case
  - Only general features are relevant
- $E = E_1 \rightarrow r > r_{\text{min}}$  $E_1 = V'(r_{\text{min}})$
  - Particle can go infinitely far

Arrive from $r = \infty$

Turning point $E = V'$  $\dot{r} = 0$

Go toward $r = \infty$

A $1/r^2$ force would make a hyperbola
Bounded Motion

\[ E = E_2 \rightarrow r_{\text{min}} < r < r_{\text{max}} \]

- Particle is confined between two circles

\[ V'(r) \]

- Goes back and forth between two radii

- Orbit may or may not be closed. (This one isn’t)

- A \(1/r^2\) force would make an ellipse
Circular Motion

- \( E = E_3 \rightarrow r = r_0 \) (fixed)
  - Only one radius is allowed

Stays on a circle

- \( E = V'(r_0) \)
- \( \dot{r} = 0 \)
- \( r = \text{const} = r_0 \)

Classification into unbounded, bounded and circular motion depends on the general shape of \( V' \)

- Not on the details (\( 1/r^2 \) or otherwise)
Another Example

\[ V = -\frac{a}{r^3}, \quad f = -\frac{3a}{r^4} \quad \Rightarrow \quad V' = -\frac{a}{r^3} + \frac{l^2}{2mr^2} \]

- **Attractive** \( r^{-4} \) **force**
  - \( V' \) has a bump
  - Particle with energy \( E \) may be either bounded or unbounded, depending on the initial \( r \)
Stable Circular Orbit

- Circular orbit occurs at the bottom of a dip of $V'$

\[
\frac{m \dot{r}^2}{2} = E - V' = 0 \quad m \ddot{r} = -\frac{dV'}{dr} = 0
\]

$r = \text{const}$

- Top of a bump works in theory, but it is unstable
  - Initial condition must be exactly
    \[\dot{r} = 0 \text{ and } r = r_0\]

Stable circular orbit requires \[\frac{d^2V'}{dr^2} > 0\]
Power Law Force

\[ V'(r) = V(r) + \frac{l^2}{2mr^2} \]

\[
\left. \frac{dV'}{dr} \right|_{r=r_0} = -f(r_0) - \frac{l^2}{mr_0^3} = 0
\]

\[
\left. \frac{d^2V'}{dr^2} \right|_{r=r_0} = -\left. \frac{df}{dr} \right|_{r=r_0} + \frac{3l^2}{mr_0^4} > 0
\]

- Suppose the force has a form \( f = -kr^n \)
  - \( k > 0 \) for attractive force
  - Condition for stable circular orbit is \(-kr_0^{n-1} < 3kr_0^{n-1}\) \(n > -3\)

Power-law forces with \( n > -3 \) can make stable circular orbit
Summary

- Started discussing Central Force Problems
  - Reduced 2-body problem into central force problem
- Problem is reduced to one equation
  - Used angular momentum conservation
- Qualitative behavior depends on
  - Unbounded, bounded, and circular orbits
  - Condition for stable circular orbits
- Next step: Can we actually solve for the orbit?