Lecture 6
Kepler Problem
(Chapter 3)
What We Did Last Time

- Discussed energy conservation
  - Defined energy function $h \leftrightarrow$ Conserved if $\frac{\partial L}{\partial t} = 0$
  - Conditions for $h = E$

- Started discussing Central Force Problems
  - Reduced 2-body problem into central force problem

- Problem is reduced to one equation
  - Used angular momentum conservation
  - Energy conservation gives

$$E = \frac{m}{2} \dot{r}^2 + \frac{1}{2} \frac{l^2}{mr^2} + V(r) = \text{const}$$

- Now we must solve this

$$m\ddot{r} = \frac{l^2}{mr^3} + f(r)$$
Goals for Today

- Analyze qualitative behavior of central-force problem
  - Solutions: bounded or unbounded
    - Determined by the “shape” of the potential
- Solve the Kepler problem
  - Get the shape of the orbit
    - As if we don’t know yet…
  - Derive Kepler’s 3rd Law
    - Period of rotation is proportional to the $3/2$ power of the major axis
Qualitative Behavior

- Integrating the radial motion isn’t always easy
  - More often impossible…

- You can still tell general behavior by looking at

  \[ \dot{r} = \sqrt{\frac{2}{m}} \left( E - V(r) - \frac{l^2}{2mr^2} \right) \]

  Quasi potential including the centrifugal force

- Energy \( E \) is conserved, and \( E - V' \) must be positive

  \[ E = \frac{m\dot{r}^2}{2} + V'(r) \]
  \[ \frac{m\dot{r}^2}{2} = E - V'(r) > 0 \]
  \[ E > V'(r) \]

- Plot \( V'(r) \) and see how it intersects with \( E \)
Inverse-Square Force

- Consider an attractive $1/r^2$ force
  
  $f(r) = -\frac{k}{r^2}$  \quad $V(r) = -\frac{k}{r}$

- Gravity or electrostatic force
  
  $V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2}$

- $1/r^2$ force dominates at large $r$
- Centrifugal force dominates at small $r$
- A dip forms in the middle
Unbounded Motion

- Take $V'$ similar to $1/r^2$ case
  - Only general features are relevant
- $E = E_1 \rightarrow r > r_{\text{min}} \quad E_1 = V'(r_{\text{min}})$
  - Particle can go infinitely far

Arrive from $r = \infty$

Turning point

$E = V' \quad \dot{r} = 0$

Go toward $r = \infty$

A $1/r^2$ force would make a hyperbola
**Bounded Motion**

\[ E = E_2 \implies r_{\text{min}} < r < r_{\text{max}} \]

- Particle is confined between two circles

Goes back and forth between two radii

Orbit may or may not be closed. (This one isn’t)

A $1/r^2$ force would make an ellipse
Circular Motion

- $E = E_3 \rightarrow r = r_0$ (fixed)
  - Only one radius is allowed
  - Stays on a circle
    - $E = V'(r_0)$
    - $\dot{r} = 0$
    - $r = \text{const} = r_0$

- Classification into unbounded, bounded and circular motion depends on the general shape of $V'$
  - Not on the details ($1/r^2$ or otherwise)
Another Example

\[
V = -\frac{a}{r^3} \quad f = -\frac{3a}{r^4}
\]

\[
V' = -\frac{a}{r^3} + \frac{l^2}{2mr^2}
\]

- Attractive \( r^{-4} \) force
  - \( V' \) has a bump
  - Particle with energy \( E \) may be either bounded or unbounded, depending on the initial \( r \)
Stable Circular Orbit

- Circular orbit occurs at the bottom of a dip of $V'$
  \[
  \frac{m\dot{r}^2}{2} = E - V' = 0 \quad \text{and} \quad m\ddot{r} = -\frac{dV'}{dr} = 0
  \]
  \[r = \text{const}\]

- Top of a bump works in theory, but it is unstable
  - Initial condition must be exactly
    \[\dot{r} = 0 \quad \text{and} \quad r = r_0\]

Stable circular orbit requires \(\frac{d^2V'}{dr^2} > 0\)
Orbit Equation

- We have been trying to solve \( r = r(t) \) and \( \theta = \theta(t) \)
- We are now interested in the shape of the orbit \( r = r(\theta) \)
- Switch from \( dt \) to \( d\theta \)

\[
\dot{l} = m r^2 \dot{\theta}
\]

\[
\frac{d}{dt} = \frac{l}{m r^2} \frac{d}{d\theta}
\]

- Switch from \( r \) to \( u \equiv 1/r \)

\[
\frac{du}{d\theta} = \frac{d}{d\theta} \left( \frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{d\theta}
\]

\[
\frac{d}{dr} = -u^2 \frac{d}{du}
\]
Solving this equation gives the shape of the orbit

- Not that it’s easy (How could it be?)
- Will do this for inverse-square force later

One more useful knowledge can be extracted without solving the equation

\[
\frac{d^2u}{d\theta^2} + u + \frac{m}{l^2} \frac{dV(\frac{1}{u})}{du} = 0
\]
Symmetry of Orbit

\[ \frac{d^2u}{d\theta^2} + u + \frac{m}{l^2} \frac{dV}{du} \left(\frac{1}{u}\right) = 0 \]

- Equation is even, or symmetric, in \( \theta \)
  - Replacing \( \theta \) with \(-\theta\) does not change the equation
  - Solution \( u(\theta) \) must be symmetric if the initial condition is
  - Choosing \( \theta = 0 \) at \( t = 0 \), \( \theta \rightarrow -\theta \) makes

\[ u(0) \rightarrow u(0) \quad \text{OK} \quad \frac{du}{d\theta}(0) \rightarrow -\frac{du}{d\theta}(0) \quad \text{OK if} \quad \frac{du}{d\theta}(0) = 0 \]

- Orbit is symmetric at angles where \( \frac{du}{d\theta} = 0 \)
Symmetry of Orbit

- Orbit is symmetric about every turning point = **apsidals**

  Orbit is invariant under reflection about apsidal vectors

  - That’s why I didn’t care too much about the sign of $\dot{r}$
  - Solve the orbit between a pair of apsidal points $\rightarrow$ Entire orbit is known

- Now it’s time to solve the equation

$$\frac{du}{d\theta} = 0$$
Solving Orbit Equation

\[ \frac{d^2 u}{d\theta^2} + u + \frac{m}{l^2} \frac{dV}{du} = 0 \]

- Integrating the diff eqn will give energy conservation
- One can use energy conservation to save effort

\[ E = \frac{mr^2}{2} + \frac{l^2}{2mr^2} + V(r) \]

\[ \dot{r} = \sqrt{\frac{2}{m} \left( E - \frac{l^2}{2mr^2} - V(r) \right)} \]

- Switch variables

\[ \frac{du}{d\theta} = -\sqrt{\frac{2mE}{l^2} - u^2 - \frac{2mV}{l^2}} \]

Integrate this…
Inverse Square Force

- Look it up in a math textbook and find

\[
\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}} = -\int d\theta
\]

- Just substitute \( \alpha, \beta \) and \( \gamma \)
- Or…
\[ \int d\theta = -\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mk}{l^2} - u^2}} = -\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4} - \left(\frac{mk}{l^2} - u\right)^2}} \]

\[ = -\frac{1}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}} \int \frac{du}{\sqrt{1 - \left(\frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}}\right)^2}} \]

Define as \( \cos \omega \)

\[ = -\int \frac{\sin \omega}{\sin \omega} d\omega = -\omega \]

\[ du = \sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}} \sin \omega d\omega \]

\[ \cos \omega = \cos(\theta - \theta') = \frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}} \]

Solve this for \( u = 1/r \)
Solution

\[ u = \frac{1}{r} = \frac{mk}{l^2} \left( 1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\theta - \theta') \right) \]

- This matches the general equation of a conic

\[ \frac{1}{r} = C \left( 1 + e \cos(\theta - \theta') \right) \]

One focus is at the origin

- \( e \) is eccentricity

<table>
<thead>
<tr>
<th>( e )</th>
<th>( E )</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e &gt; 1 )</td>
<td>( E &gt; 0 )</td>
<td>hyperbola</td>
</tr>
<tr>
<td>( e = 1 )</td>
<td>( E = 0 )</td>
<td>parabola</td>
</tr>
<tr>
<td>( e &lt; 1 )</td>
<td>( E &lt; 0 )</td>
<td>ellipse</td>
</tr>
<tr>
<td>( e = 0 )</td>
<td>( E = -\frac{mk^2}{2l^2} )</td>
<td>circle</td>
</tr>
</tbody>
</table>

Matches the qualitative classification of the orbits
Energy and Eccentricity

- $E = 0$ separates unbounded and bounded orbits
  - Borderline = Parabola
- Circular orbit requires
  $$V'(r_0) = -\frac{k}{r_0} + \frac{l^2}{2mr_0^2} = E$$
  $$\frac{dV'}{dr}\bigg|_{r_0} = \frac{k}{r_0^2} - \frac{l^2}{mr_0^3} = 0$$
  $$E = -\frac{mk^2}{2l^2}$$
Unbound Orbits

\[ \frac{1}{r} = C \left(1 + e \cos(\theta - \theta')\right) \]

- \( e > 1 \rightarrow \text{hyperbola} \)
  - \( \theta' \) is the turning point (perihelion)
  - \( \cos(\theta - \theta') > -1/e \) limits \( \theta \)
- \( e = 1 \rightarrow \text{parabola} \)
Bound Orbits

- Ends of the major axis are \( \frac{1}{r} = C(1\pm e) \)

- Length of the major axis

\[
a = 2 \left( \frac{1}{C(1+e)} + \frac{1}{C(1-e)} \right) = -\frac{k}{2E}
\]

Major axis is given by the total energy \( E \)

- Minor axis is

\[
b = a\sqrt{1-e^2} = \sqrt{-\frac{l^2}{2mE}}
\]
Rotation Period

- We know that the areal velocity is constant

\[
\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2m}
\]

- Express \( \tau \) in terms of \( a \)

\[
\tau = 2\pi \sqrt{\frac{m}{k} a^{3/2}}
\]

- Period of rotation is proportional to the 3/2 power of the major axis

Kepler’s Third Law of Planetary Motion

\[
a = -\frac{k}{2E} \quad b = \sqrt{-\frac{l^2}{2mE}}
\]

\[
A = \pi ab = \pi \sqrt{-\frac{l^2 k^2}{8mE^3}}
\]
Kepler’s Third Law

- Kepler’s third law is not exact
  - The reason: reduced mass
  - $k$ is given by the gravity
    $$f = -G \frac{Mm}{r^2} = -\frac{k}{r^2}$$
    $$k = GMm$$
  - Period of rotation becomes
    $$\tau = 2\pi \sqrt{\frac{\mu}{k}} a^{3/2} = 2\pi \sqrt{\frac{1}{G(M + m)}} a^{3/2}$$
  - Coefficient is same for all planets only if $M >> m$
So far we dealt with the shape of the orbit: $r = r(\theta)$

- We don’t have the full solutions $r = r(t)$ and $\theta = \theta(t)$

Why aren’t we doing it?

- It’s awfully complicated
  - Not that bad to get $t = t(\theta)$ \(\Rightarrow\) See Goldstein Section 3.8
  - Inverting to $\theta = \theta(t)$ impossible
  - Physicists spent *centuries* calculating approximate solutions
    - Already got physically interesting features of the solution

Leave it to the computers
Summary

- Studied qualitative behavior of the orbits
  - Bounded or unbounded ↔ Shape of $V'(r) \equiv V(r) + \frac{l^2}{2mr^2}$
- Derived orbit equation from the eqn of radial motion
  - $r$ (or $u = 1/r$) as a function of $\theta$
- Analyzed the Kepler Problem
  - Solved the orbit
  - Conic depending on $E$
  - For elliptic orbit, major axis depends only on $E$
  - Kepler’s third law of planetary motion

$$\frac{1}{r} = \frac{mk}{l^2} \left(1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\theta - \theta')\right)$$

$$a = -\frac{k}{2E}$$