What We Did Last Time

- Discussed 3-dimensional rotation
  - Preparation for rigid body motion
    - Movement in 3-d + Rotation in 3-d = 6 coordinates
- Looked for ways to describe 3-d rotation
  - Euler angles one of the many possibilities
  - Euler’s theorem
- Defined infinitesimal rotation $d\Omega$
  - Commutative (unlike finite rotation)
  - Behaves as an axial vector (like angular momentum)

$$d\Omega = n d\Phi$$
$$dr = r \times d\Omega$$
Goals For Today

- Time derivatives in rotating coordinate system
  - To calculate velocities and accelerations
  - Coriolis effect
  - Express angular velocity using Euler angles
- Try to write down Lagrangian for rigid body
  - Separate rotation from movement of CoM
  - Define inertia tensor
- Will almost get to the equation of motion

Body Coordinates

- Consider a rotating rigid body
  - Define body coordinates \((x', y', z')\)
- Between \(t\) and \(t + dt\), the body coordinates rotate by \(d\Omega = n d\Phi\)
  - The direction is CCW because we are talking about the coordinate axes
  - Observed in the space coordinates, any point \(\mathbf{r}\) on the body moves by \(d\mathbf{r} = d\Phi n \times \mathbf{r} = d\Omega \times \mathbf{r}\)
  - Sign opposite from last lecture because the rotation is CCW
General Vectors

- Now consider a general vector \( \mathbf{G} \)
  - How does it move in space/body coordinates?
  - i.e. what’s the time derivative \( \frac{d\mathbf{G}}{dt} \)?
- Movement \( d\mathbf{G} \) differs in space and body coordinates because of the rotation of the latter

\[
\left( \frac{d\mathbf{G}}{dt} \right)_{\text{space}} = \left( \frac{d\mathbf{G}}{dt} \right)_{\text{body}} + \left( \frac{d\mathbf{G}}{dt} \right)_{\text{rot}}
\]

- If \( \mathbf{G} \) is fixed to the body

\[
\left( \frac{d\mathbf{G}}{dt} \right)_{\text{body}} = 0 \quad \text{and} \quad \left( \frac{d\mathbf{G}}{dt} \right)_{\text{space}} = \mathbf{\Omega} \times \mathbf{G}
\]

\[
\left( \frac{d\mathbf{G}}{dt} \right)_{\text{rot}} = \mathbf{\Omega} \times \mathbf{G}
\]

Angular Velocity

- For any vector \( \mathbf{G} \)

\[
\left( \frac{d\mathbf{G}}{dt} \right)_{\text{space}} = \left( \frac{d\mathbf{G}}{dt} \right)_{\text{body}} + \mathbf{\Omega} \times \mathbf{G}
\]

\[
\frac{d\mathbf{G}}{dt} = \frac{d\mathbf{G}}{dt} + \mathbf{\omega} \times \mathbf{G}
\]

- \( \mathbf{\omega} = \) instantaneous angular velocity
  - Direction = \( \mathbf{n} = \) instantaneous axis of rotation
  - Magnitude = \( \frac{d\Phi}{dt} = \) instantaneous rate of rotation
- Since this works for any vector, we can say

\[
\frac{d}{dt} = \frac{d}{dt} + \mathbf{\omega} \times
\]

\[
\text{space coordinates} \quad \Rightarrow \quad \text{rotating coordinates}
\]
Coriolis Effect

 Imagine a particle observed in a rotating system
 - e.g. watching an object’s motion on Earth
 - Velocity: \( \mathbf{v}_s = \mathbf{v}_r + \mathbf{\omega} \times \mathbf{r} \)
 - Acceleration: \( \mathbf{a}_s = \left( \frac{d\mathbf{v}_s}{dt} \right)_s = \left( \frac{d\mathbf{v}_s}{dt} \right)_r + \mathbf{\omega} \times \mathbf{v}_s \)
   \[ = \mathbf{a}_r + 2\mathbf{\omega} \times \mathbf{v}_r + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]
 - Newton’s equation works in the space (inertial) system, i.e. \( \mathbf{F} = m\mathbf{a}_s \)

 \[ m\mathbf{a}_r = \mathbf{F}_{\text{eff}} = \mathbf{F} - 2m(\mathbf{\omega} \times \mathbf{v}_r) - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]

 Object appears to move according to this force

Coriolis Effect

 Last term is centrifugal force
 Middle term is Coriolis effect
 - Occurs when object is moving in the rotating frame
 - Moving objects on Earth appears to deflect toward right in the northern hemisphere
 - Hurricane wind pattern
 - Foucault pendulum

\[ \mathbf{F}_{\text{eff}} = \mathbf{F} - 2m(\mathbf{\omega} \times \mathbf{v}_r) - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]
Coriolis Effect

- Free-falling particle
  - Drop a ball and ignore air resistance
  - Coriolis effect is \(-2m(\omega \times v)\)
  - Pointing east (out of screen) \(\rightarrow x\) axis
  - Ignoring the centrifugal force

\[
\begin{align*}
    mx &= -2mv_n \sin \theta \\
    m\ddot{z} &= -mg
\end{align*}
\]

\[
\begin{align*}
    x &= \frac{\omega gt^3 \sin \theta}{3}, \\
    z &= -\frac{gt^2}{2}
\end{align*}
\]

\[
\begin{align*}
    x &= \frac{\omega \sin \theta}{3} \sqrt{\frac{(-2z)^3}{g}}
\end{align*}
\]

- For \(\sin \theta = 1\) and \(z = 100\) m \(\rightarrow x = 2.2\) cm

Euler Angles

- Use angular velocity \(\omega\) to calculate particles’ velocities
- Use Euler angles to describe the rotation of rigid bodies
- How are they connected?
  - Infinitesimal rotations can be added like vectors

\[
\omega = n_z \phi + n_x \theta + n_z \psi
\]
Euler Angles

Let’s express $\omega$ in $x’$-$y’$-$z’$

- Doing this in $x$-$y$-$z$ equally easy (or difficult)
- Must express $\mathbf{n}_z$, $\mathbf{n}_\zeta$, $\mathbf{n}_\xi$ in $x’$-$y’$-$z’$
  - $\mathbf{n}_z \rightarrow \mathbf{A}\mathbf{n}_z$, $\mathbf{n}_\zeta \rightarrow \mathbf{B}\mathbf{n}_\xi$ ($\mathbf{n}_\zeta$ is OK)

From the last lecture

\[
\mathbf{D} = \begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{C} = \begin{bmatrix}
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{B} = \begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{A} = \begin{bmatrix}
\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi & \cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi & \sin \psi \sin \theta \\
-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi & -\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi & \cos \psi \sin \theta \\
\sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta
\end{bmatrix}
\]

\[
\mathbf{n}_z = \mathbf{A} \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} = \begin{bmatrix}\sin \psi \sin \theta \\ \cos \psi \sin \theta \\ \cos \theta\end{bmatrix} \quad \mathbf{n}_\zeta = \mathbf{B} \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix} = \begin{bmatrix}\cos \psi \\ -\sin \psi \\ 0\end{bmatrix} \quad \mathbf{n}_\xi = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}
\]
Euler Angles

\[ \mathbf{\omega} = n_z \dot{\phi} + n_z \dot{\theta} + n_z \dot{\psi} = \begin{bmatrix} \dot{\phi} \sin \psi \sin \theta + \dot{\theta} \cos \psi \\ \dot{\phi} \cos \psi \sin \theta - \dot{\theta} \sin \psi \\ \dot{\phi} \cos \theta + \dot{\psi} \end{bmatrix} \]

- Remember: this is in the \( x' - y' - z' \) system
- Now we know how to express velocities in terms of time-derivatives of Euler angles
- We can write down the Lagrangian

Kinetic Energy

- Kinetic energy of multi-particle system is

\[ T = \frac{1}{2} Mv^2 + \frac{1}{2} m_i v_i^2 \]

Remember Einstein convention

- If we define the body axis from the center of mass

\[ T = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + T'(\dot{\phi}, \dot{\theta}, \dot{\psi}) \]

- \( T' \) depends only on the angular velocity
- Must be a 2\textsuperscript{nd} order homogeneous function
**Potential Energy**

- Potential energy can often be separated as well
  \[ V = V_1(x, y, z) + V_2(\phi, \theta, \psi) \]
  - Uniform gravity \( g \)  \( V_1 = -g \cdot r \)
  - Uniform magnetic field \( B \) and magnetic dipole moment \( M \)  \( V_2 = -M \cdot B \)
  - Lagrangian can then be written as
    \[ L = L_t(x, y, z, \dot{x}, \dot{y}, \dot{z}) + L_r(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) \]

It is often possible to separate the translational and rotational motions by taking the center of mass as the origin of the body coordinate axes.

**Rotational Motion**

- We concentrate on the rotational part
  - Translational part same as a single particle \( \rightarrow \) Easy
  - Consider total angular momentum \( L = m_i r_i \times v_i \)
    - \( v_i \) is given by the rotation \( \omega \) as \( v_i = \omega \times r_i \)

\[
L = m_i r_i \times (\omega \times r_i)
= m_i \left[ \omega r_i^2 - r_i (r_i \cdot \omega) \right] = \begin{bmatrix}
m_i (r_i^2 - x_i^2) & -m_i x_i y_i & -m_i x_i z_i \\
-m_i y_i x_i & m_i (r_i^2 - y_i^2) & -m_i y_i z_i \\
-m_i z_i x_i & -m_i z_i y_i & m_i (r_i^2 - z_i^2)
\end{bmatrix} \omega
\]

Inertia tensor \( I \)
Inertia Tensor

- Diagonal components are familiar moment of inertia

\[ I_{xx} = m_i (r_i^2 - x_i^2) = m_i r_i^2 \sin^2 \Theta \]

- What are the off-diagonal components?
  - \( I_{xy} \) produces \( L_y \) when the object is turned around \( x \) axis
  - Imagine turning something like:

Unbalanced one has non-zero off-diagonal components, which represents “wobbliness” of rotation

\[ \begin{bmatrix} m_i (r_i^2 - x_i^2) & -m_i x_i y_i & -m_i x_i z_i \\ -m_i y_i x_i & m_i (r_i^2 - y_i^2) & -m_i y_i z_i \\ -m_i z_i x_i & -m_i z_i y_i & m_i (r_i^2 - z_i^2) \end{bmatrix} \]

- Using \((x_i, y_i, z_i) \rightarrow (x_{i1}, x_{i2}, x_{i3})\)

\[ I_{jk} = m_i \left( r_i^2 \delta_{jk} - x_j x_k \right) \]

- We can also deal with continuous mass distribution \( \rho(r) \)

\[ I_{jk} = \int \rho(r) \left( r^2 \delta_{jk} - x_j x_k \right) dr \]
Kinetic Energy

- Kinetic energy due to rotation is \( T = \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i \)
- Using \( \mathbf{v}_i = \mathbf{\omega} \times \mathbf{r}_i \)
  \[ T = \frac{1}{2} m_i \mathbf{v}_i \cdot (\mathbf{\omega} \times \mathbf{r}_i) = \frac{\mathbf{\omega}}{2} \cdot m_i (\mathbf{r}_i \times \mathbf{v}_i) = \frac{\mathbf{\omega}}{2} \cdot \mathbf{L} \]
- Using \( \mathbf{L} = I \mathbf{\omega} \)
  \[ T = \frac{\mathbf{\omega} \cdot I \mathbf{\omega}}{2} \]
- Defining \( \mathbf{n} \) as the unit vector in the direction of \( \mathbf{\omega} \)
  \[ \mathbf{\omega} = \mathbf{\omega}_n \]
  \[ T = \frac{\mathbf{n} \cdot I \mathbf{n}}{2} \mathbf{\omega}^2 = \frac{1}{2} I \mathbf{\omega}^2 \]
  where \( I = \mathbf{n} \cdot I \mathbf{n} = m_i \left[ r_i^2 - (r_i \cdot \mathbf{n})^2 \right] \)
- NB: \( \mathbf{n} \) moves with time
- \( I = I(t) \) must change accordingly with time

Shifting Origin

- Origin of body axes does not have to be at the CoM
  - It’s convenient – Separates translational/rotational motion
- If it isn’t, \( I \) can be easily translated

\[ I = m_i (r_i \times \mathbf{n})^2 = m_i [(R + r'_i) \times \mathbf{n}]^2 \]
\[ = M (R \times \mathbf{n})^2 + m_i (r'_i \times \mathbf{n})^2 + 2m_i (R \times \mathbf{n}) \cdot (r'_i \times \mathbf{n}) \]

\( I \) of CoM \( I \) from CoM
Summary

- Found the velocity due to rotation
  - Used it to find Coriolis effect
- Connected $\boldsymbol{\omega}$ with the Euler angles
- Lagrangian $\rightarrow$ translational and rotational parts
  - Often possible if body axes are defined from the CoM
- Defined the inertia tensor
  - Calculated angular momentum and kinetic energy
- Next: Equation of motion (finally!)

\[
\frac{d}{dt} \mathbf{r} = \frac{d}{dt} \mathbf{r}_r + \boldsymbol{\omega} \times \mathbf{r}
\]

\[
\mathbf{I} = m_i \left( r_i^2 \delta_{jk} - x_{ij} x_{ik} \right)
\]

\[
\mathbf{L} = \mathbf{I} \boldsymbol{\omega}
\]

\[
T = \frac{\mathbf{n} \mathbf{n}^T}{2} \omega^2 = \frac{1}{2} \mathbf{I} \omega^2
\]