Mechanics
Physics 151

Lecture 9
Rigid Body Motion
(Chapter 4, 5)

What We Did Last Time

- Discussed 3-dimensional rotation
  - Preparation for rigid body motion
    - Movement in 3-d + Rotation in 3-d = 6 coordinates
- Looked for ways to describe 3-d rotation
  - Euler angles one of the many possibilities
  - Euler’s theorem
- Defined infinitesimal rotation \( d\Omega \)
  - Commutative (unlike finite rotation)
  - Behaves as an axial vector (like angular momentum)

Goals For Today

- Time derivatives in rotating coordinate system
  - To calculate velocities and accelerations
  - Coriolis effect
  - Express angular velocity using Euler angles
- Try to write down Lagrangian for rigid body
  - Separate rotation from movement of CoM
  - Define inertia tensor
- Will almost get to the equation of motion
Body Coordinates

- Consider a rotating rigid body
- Define body coordinates \((x', y', z')\)
- Between \(t\) and \(t + dt\), the body coordinates rotate by \(\frac{d\Phi}{dt} = \Omega\)
  - The direction is CCW because we are talking about the coordinate axes
  - Observed in the space coordinates, any point \(r\) on the body moves by \(dr = d\Phi n \times r = d\Omega \times r\)

Sign opposite from last lecture because the rotation is CCW.

General Vectors

- Now consider a general vector \(G\)
- How does it move in space/body coordinates?
- i.e. what’s the time derivative \(\frac{dG}{dt}\)?
- Movement \(\frac{dG}{dt}\) differs in space and body coordinates because of the rotation of the latter
- \((\frac{dG}{dt})_{\text{space}} = (\frac{dG}{dt})_{\text{body}} + (\frac{dG}{dt})_{\text{rot}}\) Difference is due to rotation
  - If \(G\) is fixed to the body
    \((\frac{dG}{dt})_{\text{body}} = 0\) and \((\frac{dG}{dt})_{\text{space}} = d\Omega \times G\)
  - \((\frac{dG}{dt})_{\text{rot}} = d\Omega \times G\) Generally true

Angular Velocity

- For any vector \(G\) \((\frac{dG}{dt})_{\text{space}} = (\frac{dG}{dt})_{\text{body}} + d\Omega \times G\)
- \(\omega = \text{instantaneous angular velocity}\)
  - Direction = \(n\) = instantaneous axis of rotation
  - Magnitude = \(d\Phi/dt\) = instantaneous rate of rotation
- Since this works for any vector, we can say
  \[
  \frac{d}{dt} = \frac{d}{dt} + \omega \times
  \]
  - space coordinates
  - rotating coordinates
Coriolis Effect

Imagine a particle observed in a rotating system
- e.g. watching an object’s motion on Earth
- Velocity: \( v_s = v_r + \omega \times r \)
- Acceleration: \( a_s = \frac{dv_r}{dt} = \frac{dv_r}{dt} + \omega \times v_r = a_r + 2\omega \times v_r + \omega \times (\omega \times r) \)
- Newton’s equation works in the space (inertial) system, i.e. \( F = ma \)

\[
m_a = F_{\text{eff}} = F - 2m(\omega \times v_r) - m\omega \times (\omega \times r)
\]

Object appears to move according to this force

Last term is centrifugal force
Middle term is Coriolis effect
- Occurs when object is moving in the rotating frame
- Moving objects on Earth appears to deflect toward right in the northern hemisphere
- Hurricane wind pattern
- Foucault pendulum

Free-falling particle
- Drop a ball and ignore air resistance
- Coriolis effect is \( = 2m(\omega \times v) \)
- Pointing east (out of screen) \( \rightarrow x \) axis
- Ignoring the centrifugal force

\[
\begin{align*}
mz &= -2mv \sin \theta \\
z &= \frac{mz}{-mg} \\
\end{align*}
\]

For \( \sin \theta = 1 \) and \( z = 100 \) m \( \rightarrow x = 2.2 \) cm
Euler Angles

- Use angular velocity \( \omega \) to calculate particles' velocities
- Use Euler angles to describe the rotation of rigid bodies
- How are they connected?
  - Infinitesimal rotations can be added like vectors

\[ \omega = n_z \theta + n_\xi \phi + n_s \psi \]

Let's express \( \omega \) in \( x'-y'-z' \)
- Doing this in \( x-y-z \) equally easy (or difficult)
- Must express \( n_z, n_\xi, n_s \) in \( x'-y'-z' \)
  - \( n_z \rightarrow A n_z \rightarrow B n_z \) (\( n_s \) is OK)

From the last lecture

\[
\begin{array}{c}
D & = & \left[ \begin{array}{ccc}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1 \\
\end{array} \right] \\
C & = & \left[ \begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta \\
\end{array} \right] \\
B & = & \left[ \begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1 \\
\end{array} \right] \\
A & = & \left[ \begin{array}{ccc}
\cos \phi \cos \psi - \cos \phi \sin \psi \sin \theta & \cos \phi \sin \psi + \cos \psi \sin \phi \sin \theta & \sin \phi \sin \psi + \cos \psi \cos \phi \sin \theta \\
-\sin \phi \cos \psi - \sin \phi \sin \psi \sin \theta & -\sin \phi \sin \psi + \cos \psi \cos \phi \sin \theta & \cos \phi \sin \psi + \cos \psi \cos \phi \sin \theta \\
\sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \\
\end{array} \right] \\
\end{array}
\]

\[ n_z = A n_z = B n_z = C n_z = D n_z \]
Euler Angles

- Remember: this is in the $x'-y'-z'$ system
- Now we know how to express velocities in terms of time-derivatives of Euler angles
- We can write down the Lagrangian

\[
\begin{bmatrix}
\dot{\phi}
\dot{\theta}
\dot{\psi}
\end{bmatrix}
= \begin{bmatrix}
\sin \psi \sin \theta + \dot{\theta} \cos \psi \\
\cos \psi \sin \theta - \dot{\theta} \sin \psi \\
\cos \theta + \dot{\psi}
\end{bmatrix}
\]

Kinetic Energy

- Kinetic energy of multi-particle system is

\[
T = \frac{1}{2} J \dot{\omega}^2 + \frac{1}{2} m \dot{\mathbf{v}}^2
\]

- If we define the body axis from the center of mass

\[
T' = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + T'(\dot{\phi}, \dot{\theta}, \dot{\psi})
\]

- $T'$ depends only on the angular velocity
- Must be a 2nd order homogeneous function

Potential Energy

- Potential energy can often be separated as well

\[
V = V(x,y,z) + V(\phi, \theta, \psi)
\]

- Uniform gravity $g$  $V_g = -g \cdot \mathbf{r}$
- Uniform magnetic field $\mathbf{B}$ and magnetic dipole moment $\mathbf{M}$

\[
V_M = -\mathbf{M} \cdot \mathbf{B}
\]

- Lagrangian can then be written as

\[
\mathcal{L} = \mathcal{L}(x,y,z, \dot{x}, \dot{y}, \dot{z}) + \mathcal{L}(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})
\]

It is often possible to separate the translational and rotational motions by taking the center of mass as the origin of the body coordinate axes
Rotational Motion

- We concentrate on the rotational part
  - Translational part same as a single particle → Easy
- Consider total angular momentum \( \mathbf{L} = m \mathbf{r} \times \mathbf{v} \)
  - \( \mathbf{v}_i \) is given by the rotation \( \mathbf{\omega} \) as \( \mathbf{v}_i = \mathbf{\omega} \times \mathbf{r} \)

\[
\mathbf{L} = m \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}) = m \left[ \mathbf{\omega} \mathbf{r}^2 - \mathbf{r} (\mathbf{\omega} \cdot \mathbf{r}) \right] = m \begin{bmatrix} 
  m_i (r_i^2 - x_i^2) & -m_i x_i y_i & -m_i x_i z_i \\
  -m_i y_i x_i & m_i (r_i^2 - y_i^2) & -m_i y_i z_i \\
  -m_i z_i x_i & -m_i z_i y_i & m_i (r_i^2 - z_i^2) 
\end{bmatrix} \mathbf{\omega}
\]

Inertia Tensor

- Diagonal components are familiar moment of inertia
  \[
  I_\omega = m \left( r_i^2 - x_i^2 \right) = m_i r_i^2 \sin^2 \Theta
  \]
- What are the off-diagonal components?
  - \( I_{yx} \) produces \( L_y \) when the object is turned around \( x \) axis
  - Imagine turning something like:

Balanced

Unbalanced

Unbalanced one has non-zero off-diagonal components, which represents “wobbliness” of rotation

Inertia Tensor

- Using \((x_i, y_i, z_i) \rightarrow (x_{i1}, x_{i2}, x_{i3})\)

\[
I = \begin{bmatrix} 
  m_i (r_i^2 - x_{i1}^2) & -m_i x_{i1} y_{i1} & -m_i x_{i1} z_{i1} \\
  -m_i y_{i1} x_{i1} & m_i (r_i^2 - y_{i1}^2) & -m_i y_{i1} z_{i1} \\
  -m_i z_{i1} x_{i1} & -m_i z_{i1} y_{i1} & m_i (r_i^2 - z_{i1}^2) 
\end{bmatrix}
\]

- We can also deal with continuous mass distribution \( \rho(\mathbf{r}) \)

\[
I_\mu = \int \rho(\mathbf{r}) (r^2 \delta_{ij} - x_i x_j) \, d\mathbf{r}
\]
Kinetic Energy

- Kinetic energy due to rotation is \( T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} \).
- Using \( \mathbf{v} = \mathbf{r} \times \mathbf{\omega} \), \( T = \frac{1}{2} m (\mathbf{r} \times \mathbf{\omega}) \cdot (\mathbf{r} \times \mathbf{\omega}) = \frac{1}{2} m (\mathbf{r} \times \mathbf{\omega})^2 \).
- Using \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \), \( T = \frac{1}{2} \mathbf{L} \cdot \mathbf{L} \).
- Defining \( \mathbf{n} \) as the unit vector in the direction of \( \mathbf{\omega} \), \( \mathbf{n} \times \mathbf{\omega} \) moves with time.
- \( I = I(t) \) must change accordingly with time.

Shifting Origin

- Origin of body axes does not have to be at the CoM.
- It’s convenient – separates translational/rotational motion.
- If it isn’t, \( I \) can be easily translated.

\[
\begin{align*}
I &= m (r \times n)^2 = m [(R + r') \times n]^2 \\
&= M (R \times n)^2 + m (r' \times n)^2 + 2m (r \times n) (r' \times n)
\end{align*}
\]

\( I \text{ of CoM} \quad I \text{ from CoM} \)

Summary

- Found the velocity due to rotation.
- Used it to find Coriolis effect.
- Connected \( \mathbf{\omega} \) with Euler angles.
- Lagrangian \( \rightarrow \) translational and rotational parts.
- Often possible if body axes are defined from the CoM.
- Defined the inertia tensor.
- Calculated angular momentum and kinetic energy.
- Next: Equation of motion (finally!)