What We Did Last Time

- Found the velocity due to rotation
  \[ \frac{d}{dt} \mathbf{v} = \frac{d}{dt} \mathbf{r} + \omega \times \mathbf{r} \]
- Used it to find the Coriolis effect
- Connected \( \omega \) with the Euler angles
- Lagrangian \rightarrow\ translational and rotational parts
  Often possible if body axes are defined from the CoM
- Defined the inertia tensor
- Calculated angular momentum and kinetic energy
  \[ I = I_0 \quad T = \frac{1}{2} \omega \cdot I \cdot \omega \]

Diagonalizing Inertia Tensor

- Inertia tensor \( I \) can be made diagonal
  \[ I = \mathbf{R} \mathbf{I}_0 \mathbf{R}^T \]
  \( \mathbf{R} \) is a rotation matrix

- Kinematical properties of a rigid body are fully described by its mass, principal axes, and moments of inertia
Principal Axes

- Consider a rigid body with body axes \( x\,-\,y\,-\,z \)
  - Inertia tensor \( I \) is (in general) not diagonal
  - But it can be made diagonal by \( I_{\text{new}} = RIR \)
- Rotate \( x\,-\,y\,-\,z \) by \( R \Rightarrow \) New body axes \( x'\,-\,y'\,-\,z' \)
  - In \( x'\,-\,y'\,-\,z' \) coordinates
    \[ \dot{\omega} = R\dot{\omega} \]
    \[ \ddot{L} = RL \]
    \[ = RIR \]
    \[ = RRR\omega \]
    \[ = I_{\text{new}} \omega' \]

One can choose a set of body axes that make the inertia tensor diagonal

→ Principal Axes

Finding Principal Axes

- Consider unit vectors \( \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \) along principal axes
  \[ \mathbf{n}_i = I\mathbf{n}_i \] \( \mathbf{n}_i \) is an eigenvector of \( I \) with eigenvalue \( I_i \)
- To find the principal axes and principal moments:
  - Express \( I \) in any body coordinates
  - Solve eigenvalue equation
    \[ (I - \lambda I) = 0 \]
    \[ = \lambda = I_1, I_2, I_3 \]
  - Eigenvectors point the principal axes
  - Use them to re-define the body coordinates to simplify \( I \)
  - You can often find the principal axes by just looking at the object

Rotational Equation of Motion

- Concentrate on the rotational motion
  \[ \frac{d}{dt} L = \frac{d}{dt} (\mathbf{I}) \mathbf{\omega} + \mathbf{\omega} \times L = N \]
  \[ \text{“space” axes} \]
  \[ \text{“body” axes} \]
- Take the principal axes as the body axes
  \[ L = I\omega = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \mathbf{\omega} = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix} \]
Euler’s Equation of Motion

\[
\frac{d}{dt} \begin{bmatrix}
I_1 \omega_1 \\
I_2 \omega_2 \\
I_3 \omega_3
\end{bmatrix} + \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} \times \begin{bmatrix}
I_1 \omega_1 \\
I_2 \omega_2 \\
I_3 \omega_3
\end{bmatrix} = \begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix}
\]

Euler’s equation of motion for rigid body with one point fixed

- Special cases:
  - \( \omega_3 = 0 \) \( \Rightarrow \) \( I_3 \omega_3 = N_3 \)
  - \( I_1 = I_2 \) \( \Rightarrow \) \( I_3 \omega_3 = N_3 \)

Torque-Free Motion

- No linear force \( \Rightarrow \) Conservation of linear momentum
- No torque \( \Rightarrow \) Conservation of angular momentum
- Try \( \mathbf{N} = 0 \) in Euler’s equation of motion

\[
\begin{align*}
I_1 \omega_1 - \omega_3 \omega_3 (I_1 - I_3) &= 0 \\
I_2 \omega_2 - \omega_3 \omega_3 (I_2 - I_3) &= 0 \\
I_3 \omega_3 - \omega_3 \omega_3 (I_3 - I_1) &= 0
\end{align*}
\]

Integrating these equation will give us energy and angular momentum conservation

- We will do something more intuitive (hopefully)
  - Geometrical trick by L. Poinsot

Inertia Ellipsoid

- For any direction \( \mathbf{n} \), \( I = \mathbf{n} \cdot \mathbf{n} I \)
  - If we express \( \mathbf{n} \) using principal axes \( x'-y'-z' \)

\[
I = I_{x'} = I_{y'} + I_{z'} + I_{x''}
\]

- Consider a vector \( \rho = \frac{\mathbf{n}}{\sqrt{I}} \)

\[
1 = I_{x'} \rho_x^2 = I_{y'} \rho_y^2 + I_{z'} \rho_z^2 + I_{x''} \rho_{x''}^2
\]

Inertia ellipsoid
Inertia Ellipsoid

\[ \rho = \frac{n}{\sqrt{I}} \]

Inertia ellipsoid represents the moment of inertia of a rigid body in all directions.

Usefulness of this definition will become apparent soon.

Inertia Ellipsoid

- Inertia along axis \( n \) is \( I = \textbf{n} \cdot \rho \textbf{n} \)
- \( F(\rho) = \rho \cdot L = I_1 \rho_1^2 + I_2 \rho_2^2 + I_3 \rho_3^2 = 1 \)
- \( F \) is a function (like potential) defined in the \( \rho \) space
  - \( F(\rho) = 1 \) \( \rightarrow \) Inertia ellipsoid
  - Normal of the ellipsoid given by the gradient
    \[ \nabla F = 2L \rho \]
  - Using \( \rho = \frac{n}{\sqrt{I}} = \frac{\omega}{\sqrt{2T}} \)
    \[ \nabla F = \frac{\omega}{\sqrt{2T}} L \]
    Surface of the inertia ellipsoid is perpendicular to \( L \).

Invariable Plane

- Surface of inertia ellipsoid at \( \rho \) is perpendicular to the angular momentum \( L \)
- As \( \rho \) moves, inertia ellipsoid must rotate to satisfy this condition continuously
- Consider the projection of \( \rho \) on \( L \)
  \[ \frac{\rho \cdot L}{L} = \frac{\omega L}{L} = \frac{\sqrt{2T}}{L} \]
  \[ \text{constant} \]
  - The ellipsoid is touching a fixed plane perpendicular to \( L \)
  - Invariable plane

\( L \) is conserved.

\[ \nabla F = \frac{\omega}{\sqrt{2T}} L \]
Invariable Plane

- Inertia ellipsoid touches the invariable plane.
  - Distance between the center and the plane is $\sqrt{\frac{1}{\mathbf{T}}}$.
  - Determined by the initial conditions.

- Touching point = instantaneous axis of rotation.
  - Tip of the $\mathbf{p}$ vector is momentarily at rest in space.
  - $\frac{d\mathbf{p}}{dt} = \mathbf{\omega} \times \mathbf{p} = \mathbf{\omega} \times \mathbf{\omega} = 0$
  - i.e. it’s not sliding, but rolling without slipping on the invariable plane.

= Invariable Plane

- Inertia ellipsoid rolls on the invariable plane.
  - Rotation of ellipsoid gives the rotation of the body.
  - Direction of $\mathbf{p}$ gives the direction of $\mathbf{\omega}$ in space.

- Touching point draws curves.
  - Curve drawn on the ellipsoid = polhode.
  - Curve drawn on the invariable plane = herpolhode.

- Let’s examine a few simple cases.

Simple Cases

- Inertia ellipsoid is a sphere ($I_1 = I_2 = I_3$).
  - $\mathbf{p}$ is constant and parallel to $\mathbf{L}$.
  - Stable rotation.
Simple Cases

- Initial axis is close to one of the principal axes
  - Assume $I_1 > I_2 > I_3$
  - Stable rotation around $I_1$ and $I_3$
  - Not so obvious around $I_2$
    - If $\omega_1 = \omega_3 = 0$, $\omega_2$ is constant
    - Small deviation leads to instability

Simple Cases

- Since $I_1 > I_2 > I_3$, distance $\sqrt[3]{I_1}$ allows a polhode that wraps around the inertia ellipsoid.
  - Rotation around a principal axis is stable except for the one with the intermediate moment of inertia.

Simple Cases

- Inertia ellipsoid is symmetric around one axis
  - $I_1 = I_2 = I_3$
  - $\rho$ draws a cone (space cone) on the invariable plane
  - $\rho$ draws a cone (body cone) in the inertia ellipsoid
  - Body cone rolls on the space cone
Precession

- $I_1 = I_2$ turns the Euler’s equation of motion to

$$I \omega = \alpha \omega, (I_1 - I_3)$$

$\omega = \Omega \Omega, \quad \omega = \Omega \omega$

$\Omega = \frac{I_3}{I_1 - I_3}$

$\omega$ is constant
- Consider it as a given initial condition

- $\omega$ precesses around the $I_3$ axis
- Draws the body cone

Rotation Under Torque

- We introduce torque
- Things get messy
- Consider a spinning top
- Define Euler angles

Lagrangian

- Assume $I_1 = I_2 \neq I_3$
- Kinetic energy given by
- Use Euler angles

$$T = \frac{1}{2} I_1 (\dot{\alpha_1}^2 + \dot{\alpha_3}^2) + \frac{1}{2} I_3 \dot{\alpha_3}^2$$

$$\omega = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$T = \frac{L_1}{2} (\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2) + \frac{L_2}{2} (\dot{\theta} \cos \phi + \dot{\psi} \sin \phi)^2$$
Lagrangian

- Potential energy is given by the height of the CoM
  \[ V = Mgl \cos \theta \]
- Lagrangian is
  \[ L = \frac{I_{1}}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_{2}}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]
- Finally we are in real business!
- How we solve this?
  - Note \( \phi \) and \( \psi \) are cyclic
  - Can define conjugate momenta that conserve
- To be continued …

Summary

- Discussed rotational motion of rigid bodies
  - Euler’s equation of motion
- Analyzed torque-free rotation
  - Introduced the inertia ellipsoid
  - It rolls on the invariant plane
  - Dealt with simple cases
- Started discussing heavy top
  - Found Lagrangian \( \rightarrow \) Will solve this next time
  \[ L = \frac{I_{1}}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_{2}}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]