Lecture 11
Rigid Body Motion
(Chapter 5)
Please fill out the midterm evaluation form

- Critical feedback for me
  - to evaluate how well (or badly) I’m teaching
  - to adjust the level of the course according to your needs
  - to receive your suggestions for improvements
- Be critical, and be specific
  - I can’t fix them if you don’t tell me what’s wrong
- It’s anonymous and confidential

Thank you!
What We Did Last Time

- Discussed rotational motion of rigid bodies
  - Euler’s equation of motion
- Analyzed torque-free rotation
  -Introduced the inertia ellipsoid
  -It rolls on the invariant plane
  -Dealt with simple cases
- Started discussing heavy top
  -Found Lagrangian \( \rightarrow \) Analyze it today

\[
L = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta
\]
Heavy Top

- Top is spinning on a fixed point
- Lagrangian is
  \[ L = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]
- \( \phi \) and \( \psi \) are cyclic
  - Symmetry
- \( p_\phi \) and \( p_\psi \) are conserved
Conserved Momenta

\[ L = \frac{I_1}{2} (\dot{\theta}^2 + \phi^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\phi} \cos \theta + \dot{\psi}) = I_3 \omega_3 = \text{const.} \equiv I_1 a \]

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 \cos \theta (\dot{\phi} \cos \theta + \dot{\psi}) = \text{const.} \equiv I_1 b \]

- Solve them for \( \dot{\phi} \) and \( \dot{\psi} \)

\[ \dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta} \quad \dot{\psi} = \frac{I_1 a}{I_3} - \cos \theta \frac{b - a \cos \theta}{\sin^2 \theta} \]

- We need \( \theta(t) \) to get \( \phi(t) \) and \( \psi(t) \)

Got rid of 2 degrees of freedom
Energy Conservation

Middle term is $\frac{1}{2} I_3 \omega_3^2$

We’ve got a 1-dim equation of motion of $\theta$

It looks like a particle of “mass” $I_1$
under a potential

$$V(\theta) = \frac{I_1}{2} \left( \frac{b - a \cos \theta}{\sin \theta} \right)^2 + Mgl \cos \theta$$
Simplify the equation of motion by defining

\[
\alpha \equiv \frac{2E - I_3 \omega_3^2}{I_1} \quad \text{and} \quad \beta \equiv \frac{2Mgl}{I_1}
\]

\[
\text{EoM becomes} \quad \alpha = \dot{\theta}^2 + \left( \frac{b - a \cos \theta}{\sin \theta} \right)^2 + \beta \cos \theta
\]

Switch variable from \( \theta \) to \( u = \cos \theta \)

\[
\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2
\]

Integrate

\[
t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^2}}
\]

Elliptic integral
Try to extract qualitative behavior

- Same way as we did with central force problem

Consider the RHS of the last equation

\[ \dot{u}^2 = f(u) \equiv (1-u^2)(\alpha - \beta u) - (b - au)^2 \]

\[ = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \]

- Physical range is \( f(u) = \dot{u}^2 \geq 0 \) and \(-1 \leq u \leq 1\)

- \( f(u) \) is a cubic function of \( u \) with \( \beta \equiv \frac{2Mgl}{I_1} > 0 \)

- \( f(\pm 1) = -(b - au)^2 \leq 0 \)

These conditions constrain the shape of \( f(u) \)
Shape of $f(u)$

- $f(u) = 0$ has 3 roots $-1 \leq u_1 \leq u_2 \leq 1 \leq u_3$

- Solution for $u^2 = f(u)$ is bounded inside $u_1 \leq u \leq u_2$

- $\theta$ oscillates between $\arccos(u_1)$ and $\arccos(u_2)$

- $\phi$ and $\psi$ determined by

\[ \dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta} \]

\[ \dot{\psi} = \frac{I_1 a}{I_3} - \cos \theta \frac{b - a \cos \theta}{\sin^2 \theta} \]
Nutation

Consider the sign of \( \dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta} = \frac{b - au}{1 - u^2} \)

- \( \dot{\phi} \) changes sign at \( u = u' = b/a \)
  - \( u' < u_1 \) or \( u' > u_2 \)
  - \( u_1 < u' < u_2 \)

- \( \phi \) is monotonous
- \( \phi \) switches direction

locus
Initial Condition

- Suppose the figure axis is initially at rest
  - Spin the top, then release it “quietly”
  - \( \dot{\theta}_{t=0} = 0 \) \( \Rightarrow \) \( f(u_{t=0}) = 0 \) \( \Rightarrow \) \( u_{t=0} = u_1 \) or \( u_2 \)
  - \( \dot{\phi}_{t=0} = 0 \) \( \Rightarrow \) \( b - au_{t=0} = 0 \) \( \Rightarrow \) \( u_{t=0} = u' \)

- Initially, the figure axis falls
- It then picks up precession in \( \phi \)
- How does it know which way to go?
Origin of Precession

- Angular momentum conservation

\[ p_\psi = \frac{\partial L}{\partial \psi} = I_3 \omega_3 \]
\[ p_\phi = \frac{\partial L}{\partial \phi} = I_1 \dot{\phi} \sin^2 \theta + I_3 \omega_3 \cos \theta \]

- \( \omega_3 \) is constant
- As the figure axis falls, \( \omega_3 \)’s contribution to \( p_\phi \) decreases
- \( \phi \) must start precessing to make up for it
- Direction of precession is same as that of spin
Uniform Precession

Can we make a top precess without bobbing?

- i.e. $\dot{\theta} = 0, \dot{\phi} = \text{const}$
- We need to have a double root for $f(u) = 0$

$$f(u_0) = (1-u_0^2)(\alpha - \beta u_0) - (b - au_0)^2 = 0$$

$$f'(u_0) = -2u_0(\alpha - \beta u_0) - \beta(1-u_0^2) + 2a(b - au_0) = 0$$

Combine

$$\beta = \frac{a\dot{\phi} - \dot{\phi}^2 u_0}{2}$$

$I_1 a \equiv I_3 \omega_3$

$$\beta \equiv \frac{2Mgl}{I_1}$$

$$Mgl = \dot{\phi}(I_3 \omega_3 - I_1 \dot{\phi} \cos \theta_0)$$

$$\dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta}$$
**Uniform Precession**

\[ Mgl = \dot{\phi}(I_3 \omega_3 - I_1 \dot{\phi} \cos \theta_0) \]

- For any given value of \( \omega_3 \) and \( \cos \theta_0 \), you must give exactly the right “push” in \( \phi \) to achieve uniform precession.
- Quadratic equation \( \rightarrow \) 2 solutions
  - Same top can do “fast” or “slow” precession
- For the solutions to exist \( I_3^2 \omega_3^2 > 4MglI_1 \cos \theta_0 \)

\[ \omega_3 > \frac{2}{I_3} \sqrt{MglI_1 \cos \theta_0} \]

- Uniform precession is achieved only by a fast top.
**Magnetic Dipole Moment**

- Consider a rigid body made of charged particles
  - Mass $m_i$, charge $q_i$, position $\mathbf{r}_i$, velocity $\mathbf{v}_i$
- If there is uniform magnetic field $\mathbf{B}$
  - Each particle feels force $\mathbf{F}_i = q_i \mathbf{v}_i \times \mathbf{B}$
  - If CoM is at rest and $q_i/m_i = \text{const}$
    
    $\sum_i q_i \mathbf{r}_i = \text{const}$

    $\mathbf{F} = \sum_i q_i \mathbf{v}_i \times \mathbf{B} = \frac{q}{m} \sum_i m_i \mathbf{v}_i \times \mathbf{B} = 0$

    No net force

- How about the torque?

    $\mathbf{N} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i q_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{B}) = \frac{q}{m} \sum_i m_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{B})$

    No sum over $i$!
Magnetic Dipole Moment

- Using \( \mathbf{v}_i = \omega \times \mathbf{r}_i \)

\[
N = \frac{q}{m} m_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{B}) = \frac{q}{m} m_i (\omega \times \mathbf{r}_i)(\mathbf{r}_i \cdot \mathbf{B})
\]

- Explicit calculation using polar coordinates

\[
(\omega \times \mathbf{r}_i)(\mathbf{r}_i \cdot \mathbf{B}) = \omega r_i^2 B \sin \theta \begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix} (\sin \theta \cos \phi \sin \Theta + \cos \theta \cos \Theta)
\]

- Take time average \( \Rightarrow \) Assume rotation is fast

\[
N = \frac{q}{2m} m_i (r_i \sin \theta)^2 \omega \times \mathbf{B} = \frac{q}{2m} \mathbf{L} \times \mathbf{B}
\]
Magnetic dipole moment

- Magnetic dipole \( \mathbf{M} \) in \( \mathbf{B} \) feels the torque \( \mathbf{N} = \mathbf{M} \times \mathbf{B} \)
- Fast spinning charged rigid body has a magnetic moment \( \mathbf{M} = \gamma \mathbf{L} \)
  \[ \gamma = \frac{q}{2m} \]  
  gyromagnetic ratio

- Equation of motion \( \frac{d\mathbf{L}}{dt} = \gamma \mathbf{L} \times \mathbf{B} \)
  - This makes \( \mathbf{L} \) to precess around \( \mathbf{B} \)
  - Angular velocity of precession is \( \omega_{\text{precess}} = -\gamma \mathbf{B} = -\frac{q}{2m} \mathbf{B} \)
  
Larmor frequency
Elementary Particles

- Particles such as electrons or protons have
  - Spin, or intrinsic angular momentum, \( s \)
  - Magnetic moment \( \mu \)
- Dirac equation for a spin-1/2 particle predict
  - Differs from classical charged object by factor 2
  - Particle physicists say \( \mu = \frac{g q}{2m} s \) \( g = \begin{cases} 1 & \text{classical object} \\ 2 & \text{Dirac particle} \end{cases} \)
  - \( g = 2 \) for electron and muon \( \leftrightarrow \) Dirac particles
  - \( g = 2.8 \) for proton, \(-1.9\) for neutron \( \leftrightarrow \) composite particles
Anomalous Magnetic Moment

- $\mu$ of electron and muon known very accurately

$$g_{\text{electron}} = 2.002319304374 \pm 0.000000000008$$
$$g_{\text{muon}} = 2.002331832 \pm 0.0000000012$$

- Not pure Dirac particles, but surrounded by thin cloud of virtual particles due to quantum fluctuation

- Measurement uses spin precession
  - Store particles with known spin orientation in B field
  - Measure spin direction after time $t$

$$\omega_{\text{precess}} = -\frac{gq}{2m}B$$

Need to know B very accurately
Muon g–2 Experiment

BNL E-821 muon storage ring

\( g_{\mu} = 2.0023318404 \pm 0.0000000030 \)
Summary

- Analyzed the motion of a heavy top
  - Reduced into 1-dimensional problem of $\theta$
  - Qualitative behavior $\Rightarrow$ Precession + nutation
  - Initial condition vs. behavior
- Magnetic dipole moment of spinning charged object
  - $\mathbf{M} = \gamma \mathbf{L}$, where $\gamma = q/2m$ is the gyromagnetic ratio
  - $\mathbf{L}$ precesses in magnetic field by $\omega = -\gamma \mathbf{B}$
  - $\gamma$ of elementary particles contains interesting physics
- Done with rigid bodies
  - Next: Oscillation