Mechanics
Physics 151

Lecture 11
Rigid Body Motion
(Chapter 5)

Administrivia

- Please fill out the midterm evaluation form
  - Critical feedback for me
    - to evaluate how well (or badly) I’m teaching
    - to adjust the level of the course according to your needs
    - to receive your suggestions for improvements
  - Be critical, and be specific
    - I can’t fix them if you don’t tell me what’s wrong
  - It’s anonymous and confidential
- Thank you!
What We Did Last Time

- Discussed rotational motion of rigid bodies
  - Euler’s equation of motion
- Analyzed torque-free rotation
  - Introduced the inertia ellipsoid
  - It rolls on the invariant plane
  - Dealt with simple cases
- Started discussing heavy top
  - Found Lagrangian \( \mathcal{L} \) \( \rightarrow \) Analyze it today

\[
\mathcal{L} = \frac{I_1}{2} \left( \dot{\phi}^2 + \phi^2 \sin^2 \theta \right) + \frac{I_3}{2} \left( \dot{\phi} \cos \theta + \dot{\psi} \right)^2 - Mgl \cos \theta
\]

Heavy Top

- Top is spinning on a fixed point
- Lagrangian is

\[
\mathcal{L} = \frac{I_1}{2} \left( \dot{\phi}^2 + \phi^2 \sin^2 \theta \right) + \frac{I_3}{2} \left( \dot{\phi} \cos \theta + \dot{\psi} \right)^2 - Mgl \cos \theta
\]

- \( \phi \) and \( \psi \) are cyclic
- Symmetry
- \( p_\phi \) and \( p_\psi \) are conserved
Conserved Momenta

\[ L = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_3 (\dot{\phi} \cos \theta + \dot{\psi}) = I_3 \omega_3 = \text{const.} \equiv I_1 a \]

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 \cos \theta (\dot{\phi} \cos \theta + \dot{\psi}) = \text{const.} \equiv I_1 b \]

- Solve them for \( \dot{\phi} \) and \( \psi \)
  \[ \dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta} \quad \psi = \frac{I_1 a}{I_3} - \cos \theta \frac{b - a \cos \theta}{\sin^2 \theta} \]

- We need \( \theta(t) \) to get \( \phi(t) \) and \( \psi(t) \)

Got rid of 2 degrees of freedom

Energy Conservation

\[ E = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 + Mgl \cos \theta \]

- Middle term is \( \frac{1}{2} I_3 \omega_3^2 \)

\[ E' = E - \frac{1}{2} I_3 \omega_3^2 = \frac{I_1}{2} \dot{\theta}^2 + \frac{1}{2} \frac{(b - a \cos \theta)^2}{\sin^2 \theta} + Mgl \cos \theta \]

- We’ve got a 1-dim equation of motion of \( \theta \)
  - It looks like a particle of “mass” \( I_1 \) under a potential

\[ V(\theta) = \frac{I_1}{2} \left( \frac{b - a \cos \theta}{\sin \theta} \right)^2 + Mgl \cos \theta \]
1-D Equation of Motion

- Simplify the equation of motion by defining
  \[ \alpha \equiv \frac{2E - I_3 \omega_3^2}{I_1} \quad \text{and} \quad \beta \equiv \frac{2Mgl}{I_1} \]

  \[ \text{EoM becomes} \quad \alpha = \dot{\theta}^2 + \left( \frac{b - a \cos \theta}{\sin \theta} \right)^2 + \beta \cos \theta \]

  - Switch variable from \( \theta \) to \( u = \cos \theta \)

  \[ \text{EoM} \implies \dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \]

  - Integrate
  \[ t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^2}} \]

  Elliptic integral

Qualitative Behavior

- Try to extract qualitative behavior
  - Same way as we did with central force problem

- Consider the RHS of the last equation
  \[ \dot{u}^2 = f(u) \equiv (1-u^2)(\alpha - \beta u) - (b - au)^2 \]
  \[ = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \]

  - Physical range is \( f(u) = \dot{u}^2 \geq 0 \) and \( -1 \leq u \leq 1 \)

  - \( f(u) \) is a cubic function of \( u \) with \( \beta \equiv \frac{2Mgl}{I_1} > 0 \)

  - \( f(\pm 1) = -(b - au)^2 \leq 0 \)

  These conditions constrain the shape of \( f(u) \)
Shape of $f(u)$

- $f(u) = 0$ has 3 roots $-1 \leq u_1 \leq u_2 \leq u_3$
- Solution for $\dot{u}^2 = f(u)$ is bounded inside $u_i \leq u \leq u_2$
- $\theta$ oscillates between $\arccos(u_1)$ and $\arccos(u_2)$
- $\phi$ and $\psi$ determined by

\[
\dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta} \\
\dot{\psi} = \frac{I_1 a}{I_3} - \cos \theta \frac{b - a \cos \theta}{\sin^2 \theta}
\]

Nutation

- Consider the sign of $\dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta} = \frac{b - au}{1 - u^2}$
- $\dot{\phi}$ changes sign at $u = u' = b / a$
  - $u' < u_1$ or $u' > u_2$ $\implies$ $\phi$ is monotonous
  - $u_1 < u' < u_2$ $\implies$ $\phi$ switches direction

locus
Initial Condition

- Suppose the figure axis is initially at rest
  - Spin the top, then release it “quietly”
  - $\dot{\theta}_{t=0} = 0 \implies f(u_{t=0}) = 0 \implies u_{t=0} = u_1 \text{ or } u_2$
  - $\phi_{t=0} = 0 \implies b - au_{t=0} = 0 \implies u_{t=0} = u'$

- Initially, the figure axis falls
- It then picks up precession in $\phi$
- How does it know which way to go?

Origin of Precession

- Angular momentum conservation
  \[
  p_\psi = \frac{\partial L}{\partial \psi} = I_3 \omega_3 \\
  p_\phi = \frac{\partial L}{\partial \phi} = I_3 \omega_3 \sin^2 \theta + I_4 \omega_3 \cos \theta
  \]
  - $\omega_3$ is constant
  - As the figure axis falls, $\omega_3$’s contribution to $p_\phi$ decreases
  - $\phi$ must start precessing to make up for it
  - Direction of precession is same as that of spin
Uniform Precession

- Can we make a top precess without bobbing?
  - i.e. $\dot{\theta} = 0, \dot{\phi} = \text{const}$
  - We need to have a double root for $f(u) = 0$

\[
f(u_0) = (1 - u_0^2)(\alpha - \beta u_0) - (b - au_0)^2 = 0
\]
\[
f'(u_0) = -2u_0(\alpha - \beta u_0) - \beta(1 - u_0^2) + 2a(b - au_0) = 0
\]

Combine

\[
\beta = \frac{2a(\dot{\phi} - \phi^2 u_0)}{2}
\]

\[
I, a \equiv I, \omega_3
\]

\[
\beta \equiv \frac{2Mgl}{I}
\]

\[
Mgl = \dot{\phi}(I, \omega_3 - I, \dot{\phi} \cos \theta_0)
\]

\[
\dot{\phi} = \frac{b - a \cos \theta}{\sin^2 \theta}
\]

Uniform Precession

- For any given value of $\omega_3$ and $\cos \theta_0$, you must give exactly the right “push” in $\phi$ to achieve uniform precession
- Quadratic equation $\Rightarrow$ 2 solutions
  - Same top can do “fast” or “slow” precession
  - For the solutions to exist $I^2 \omega_3^2 > 4MglI_1 \cos \theta_0$

\[
\omega_3 > \frac{2}{I_3} \sqrt{MglI_1 \cos \theta_0}
\]

- Uniform precession is achieved only by a fast top
Magnetic Dipole Moment

- Consider a rigid body made of charged particles
  - Mass \(m_i\), charge \(q_i\), position \(\mathbf{r}_i\), velocity \(\mathbf{v}_i\)
- If there is uniform magnetic field \(\mathbf{B}\)
  - Each particle feels force \(\mathbf{F}_i = q_i \mathbf{v}_i \times \mathbf{B}\)
  - If CoM is at rest and \(q_i/m_i = \text{const}\)
    - No sum over \(i\)!
  - No net force

\[
\mathbf{F} = q_i \mathbf{v}_i \times \mathbf{B} = \frac{q}{m} m_i \mathbf{v}_i \times \mathbf{B} = 0
\]

- How about the torque?

\[
\mathbf{N} = \mathbf{r}_i \times \mathbf{F}_i = q_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{B}) = \frac{q}{m} m_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{B})
\]

Magnetic Dipole Moment

- Using \(\mathbf{v}_i = \omega \times \mathbf{r}_i\)

\[
\mathbf{N} = \frac{q}{m} m_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{B}) = \frac{q}{m} m_i \omega \times r_i \times (\mathbf{v}_i \times \mathbf{B})
\]

- Explicit calculation using polar coordinates

\[
(\omega \times \mathbf{r}_i)(\mathbf{r}_i \cdot \mathbf{B}) = \omega r_i^2 B \sin \theta \begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix} (\sin \theta \cos \phi \sin \Theta + \cos \theta \cos \Theta)
\]

- Take time average \(\Rightarrow\) Assume rotation is fast

\[
\mathbf{N} = \frac{q}{2m} m_i \left(r_i \sin \theta \right)^2 \omega \times \mathbf{B} = \frac{q}{2m} \mathbf{L} \times \mathbf{B}
\]
Magnetic Dipole Moment

- Magnetic dipole \( \mathbf{M} \) in \( \mathbf{B} \) feels the torque \( \mathbf{N} = \mathbf{M} \times \mathbf{B} \)
- Fast spinning charged rigid body has a magnetic moment \( \mathbf{M} = \gamma \mathbf{L} \)
  \[ \gamma = \frac{q}{2m} \]  
  gyromagnetic ratio
- Equation of motion \( \frac{d\mathbf{L}}{dt} = \gamma \mathbf{L} \times \mathbf{B} \)
  - This makes \( \mathbf{L} \) to precess around \( \mathbf{B} \)
  - Angular velocity of precession is
  \[ \omega_{\text{precess}} = -\gamma \mathbf{B} = -\frac{q}{2m} \mathbf{B} \]
  Larmor frequency

Elementary Particles

- Particles such as electrons or protons have
  - Spin, or intrinsic angular momentum, \( s \)
  - Magnetic moment \( \mu \)
- Dirac equation for a spin-1/2 particle predict \( \mu = \frac{q}{m} s \)
  - Differs from classical charged object by factor 2
  - Particle physicists say \( \mu = \frac{gq}{2m} s \)  
    \[ g = \begin{cases} 1 & \text{classical object} \\ 2 & \text{Dirac particle} \end{cases} \]
  - \( g = 2 \) for electron and muon \( \leftrightarrow \) Dirac particles
  - \( g = 2.8 \) for proton, \(-1.9\) for neutron \( \leftrightarrow \) composite particles
Anomalous Magnetic Moment

- $\mu$ of electron and muon known very accurately
  - $g_{\text{electron}} = 2.002319304374 \pm 0.000000000008$
  - $g_{\text{muon}} = 2.002331832 \pm 0.0000000012$

- Not pure Dirac particles, but surrounded by thin cloud of virtual particles due to quantum fluctuation

- Measurement uses spin precession
  - Store particles with known spin orientation in B field
  - Measure spin direction after time $t$

\[
\omega_{\text{precess}} = -\frac{g q}{2m} B
\]

Need to know B very accurately

Muon $g$–2 Experiment

BNL E-821 muon storage ring
g$_{\text{muon}} = 2.0023318404 \pm 0.000000030$
Summary

- Analyzed the motion of a heavy top
  - Reduced into 1-dimensional problem of $\theta$
  - Qualitative behavior $\rightarrow$ Precession + nutation
  - Initial condition vs. behavior
- Magnetic dipole moment of spinning charged object
  - $\mathbf{M} = \gamma \mathbf{L}$, where $\gamma = q/2m$ is the gyromagnetic ratio
  - $\mathbf{L}$ precesses in magnetic field by $\mathbf{\omega} = -\gamma \mathbf{B}$
  - $\gamma$ of elementary particles contains interesting physics
- Done with rigid bodies
  - Next: Oscillation