Mechanics
Physics 151

Lecture 11
Rigid Body Motion
(Chapter 5)

Administrivia

- Please fill out the midterm evaluation form
- Critical feedback for me
  - to evaluate how well (or badly) I’m teaching
  - to adjust the level of the course according to your needs
  - to receive your suggestions for improvements
- Be critical, and be specific
  - I can’t fix them if you don’t tell me what’s wrong
  - It’s anonymous and confidential
- Thank you!

What We Did Last Time

- Discussed rotational motion of rigid bodies
  - Euler’s equation of motion
- Analyzed torque-free rotation
  - Introduced the inertia ellipsoid
  - It rolls on the invariant plane
  - Deal with simple cases
- Started discussing heavy top
  - Found Lagrangian \(\mathcal{L}\) Analyze it today
  \[
  \mathcal{L} = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_1}{2}(\phi \cos \theta + \dot{\psi})^2 - Mg\ell \cos \theta
  \]
**Heavy Top**

- Top is spinning on a fixed point
- Lagrangian is
  \[ L = \frac{I}{2} (\dot{\phi}^2 + \phi^2 \sin^2 \theta) + \frac{I_1}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]
- \( \phi \) and \( \psi \) are cyclic
- Symmetry
- \( p_\phi \) and \( p_\psi \) are conserved

![Diagram of a heavy top](image)

**Conserved Momenta**

\[ L = \frac{I}{2} (\dot{\phi}^2 + \phi^2 \sin^2 \theta) + \frac{I_1}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgl \cos \theta \]

\[ p_\phi = \frac{dL}{d\dot{\phi}} = I_1 (\dot{\phi} \cos \theta + \dot{\psi}) = I_\phi = \text{const.} = I \]

\[ p_\psi = \frac{dL}{d\dot{\psi}} = I_1 \dot{\psi} + I_\psi = \text{const.} = I \]

- Solve them for \( \dot{\phi} \) and \( \dot{\psi} \)
- \( \dot{\phi} = \frac{b-a \cos \theta}{\sin \theta} \)
- \( \dot{\psi} = \frac{I_\psi}{I_1} - \frac{b-a \cos \theta}{\sin \theta} \)

- We need \( \dot{\theta}(t) \) to get \( \dot{\phi}(t) \) and \( \dot{\psi}(t) \)

Got rid of 2 degrees of freedom

**Energy Conservation**

\[ E = \frac{I}{2} (\dot{\phi}^2 + \phi^2 \sin^2 \theta) + \frac{I_1}{2} (\dot{\phi} \cos \theta + \dot{\psi})^2 + Mgl \cos \theta \]

- Middle term is \( \frac{1}{2} I \dot{\theta}^2 \)
- \( E' = E - \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} I_1 (b-a \cos \theta)^2 + Mgl \cos \theta \)

- We’ve got a 1-dim equation of motion of \( \theta \)
  - It looks like a particle of “mass” \( I_1 \)
  under a potential
  \[ V(\theta) = \frac{I_1}{2} \left( \frac{b-a \cos \theta}{\sin \theta} \right)^2 + Mgl \cos \theta \]
1-D Equation of Motion

- Simplify the equation of motion by defining
  \[ \alpha = \frac{2E - I_1 a^2}{I_1} \quad \text{and} \quad \beta = \frac{2MgI_1}{I_1} \]

  \[ \text{EqM becomes} \quad \alpha = \dot{\theta}^2 + \left( \frac{b - a \cos \theta}{\sin \theta} \right)^2 + \beta \cos \theta \]

- Switch variable from \( \theta \) to \( u = \cos \theta \)

  \[ \text{EqM} \quad \ddot{u} = (1 - u^2)(\alpha - \beta u) - (b - au)^2 \]

- Integrate
  \[ t = \int_{u_1}^{u_2} \frac{du}{\sqrt{(1 - u^2)(\alpha - \beta u) - (b - au)^2}} \]

  Elliptic integral

Qualitative Behavior

- Try to extract qualitative behavior
  - Same way as we did with central force problem
  - Consider the RHS of the last equation
    \[ \ddot{u} = f(u) = (1 - u^2)(\alpha - \beta u) - (b - au)^2 \]
    \[ = \beta u^2 - (\alpha + a^2)u + (2ab - \beta)u + (\alpha - b^2) \]

  - Physical range is \( f(u) = \ddot{u} \geq 0 \) and \(-1 \leq u \leq 1\)

  - \( f(u) \) is a cubic function of \( u \) with \( \beta = \frac{2Ma}{I_1} > 0 \)

  \[ f(\pm 1) = -(b - au)^2 \leq 0 \]

  These conditions constrain the shape of \( f(u) \)

Shape of \( f(u) \)

- \( f(u) = 0 \) has 3 roots \(-1 \leq u_1 < u_2 \leq 1\)
  - Solution for \( \ddot{u} = f(u) \) is bounded inside \( u_1 \leq u \leq u_2 \)

- \( \theta \) oscillates between \( \arccos(u_1) \) and \( \arccos(u_2) \)

  - \( \phi \) and \( \psi \) determined by
    \[ \phi = \frac{b - a \cos \theta}{\sin^2 \theta} \]
    \[ \psi = \frac{1, a - \cos \theta}{\sin \theta} \frac{b - a \cos \theta}{\sin \theta} \]
Nutation

Consider the sign of \[ \dot{\theta} = \frac{b - a \cos \theta}{\sin \theta} \frac{b - au}{1 - u'} \]
- \( u' < u_1 \) or \( u' > u_2 \)
- \( \dot{\theta} = u' = b/a \) is monotonous
- \( u_1 < u' < u_2 \)
- \( \dot{\theta} \) switches direction

Initial Condition

Suppose the figure axis is initially at rest
- Spin the top, then release it “quietly”
- \( \theta_{ix} = 0 \) or \( u_{ix} = u_1 \) or \( u_2 \)
- \( \phi_{ix} = 0 \) or \( b - au_{ix} = 0 \) or \( u_{ix} = u' \)

Initially, the figure axis falls
- It then picks up precession in \( \phi \)
- How does it know which way to go?

Origin of Precession

Angular momentum conservation

\[ p_\phi = \frac{\partial L}{\partial \phi} = I \dot{\phi} \]
\[ p_\theta = \frac{\partial L}{\partial \theta} = I \dot{\phi} \sin^2 \theta + I \dot{\phi} \cos \theta \]
- \( \phi_0 \) is constant
- As the figure axis falls, \( \phi_0 \)'s contribution to \( p_\phi \) decreases
- \( \phi \) must start precessing to make up for it
- Direction of precession is same as that of spin
**Uniform Precession**

- Can we make a top precess without bobbing?
  - i.e., $\dot{\theta} = 0, \dot{\phi} = \text{const}$
  - We need to have a double root for $f(u) = 0$
    
    \[
    f(u) = (1-u^2)(\alpha - \beta u) - (b-\alpha u) = 0
    \]
    
    \[
    f'(u) = -2\alpha u - \beta (1-u^2) - \beta (1- u^2) + 2\alpha (b-\alpha u) = 0
    \]
    
    Combine $\beta = a\phi - \dot{\phi} u$

- For any given value of $\omega$ and $\cos \theta_0$, you must give exactly the right “push” in $\phi$ to achieve uniform precession

- Quadratic equation $\Rightarrow$ 2 solutions
  - Same top can do “fast” or “slow” precession
  - For the solutions to exist
    
    \[
    \omega_0 > \frac{2}{I_1} \sqrt{MgL \cos \theta_0}
    \]

- Uniform precession is achieved only by a fast top

**Magnetic Dipole Moment**

- Consider a rigid body made of charged particles
  - Mass $m_i$, charge $q_i$, position $r_i$, velocity $v_i$
  - If there is uniform magnetic field $B$
    
    - Each particle feels force $F_i = q_i v_i \times B$
    - If CoM is at rest and $q/m_i = \text{const}$
      
      $F = q \times B = \frac{q}{m} m_i v_i \times B = 0$
      
      No net force

- How about the torque?
  
  \[
  N = \mathbf{r} \times F = q \mathbf{r} \times (v \times B) = \frac{q}{m} m_i \mathbf{r} \times (v \times B)
  \]
Magnetic Dipole Moment

- Using \( \mathbf{V}_i = \mathbf{\omega} \times \mathbf{r} \)

\[
N = \frac{q}{m} m \mathbf{r} \times (\mathbf{v} \times \mathbf{B}) = \frac{q}{m} m (\mathbf{\omega} \times \mathbf{r}) (\mathbf{r} \cdot \mathbf{B})
\]

- Explicit calculation using polar coordinates

\[
(\mathbf{\omega} \times \mathbf{r}) (\mathbf{r} \cdot \mathbf{B}) = \omega \varepsilon^2 B \sin \theta \begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix} (\sin \theta \cos \phi \sin \Theta + \cos \theta \cos \Theta)
\]

- Take time average → Assume rotation is fast

\[
\mathbf{N} = \frac{q}{2m} m (\mathbf{r} \cdot \mathbf{r} \sin \Theta) \mathbf{\omega} \times \mathbf{B} = \frac{q}{2m} \mathbf{L} \times \mathbf{B}
\]

Magnetic Dipole Moment

- Magnetic dipole \( \mathbf{M} \) in \( \mathbf{B} \) feels the torque \( \mathbf{N} = \mathbf{M} \times \mathbf{B} \)

- Fast spinning charged rigid body has a magnetic moment \( \mathbf{M} = \gamma \mathbf{L} \)

\[
\gamma = \frac{q}{2m}
\]

- Gyromagnetic ratio

- Equation of motion \( \frac{d\mathbf{L}}{dt} = \gamma \mathbf{L} \times \mathbf{B} \)

- This makes \( \mathbf{L} \) to precess around \( \mathbf{B} \)

- Angular velocity of precession is \( \omega_{\text{precess}} = -\gamma B = -\frac{q}{2m} B \)

Elementary Particles

- Particles such as electrons or protons have
  - Spin, or intrinsic angular momentum, \( \mathbf{s} \)
  - Magnetic moment \( \mathbf{\mu} \)

- Dirac equation for a spin-1/2 particle predict \( \mathbf{\mu} = \frac{q}{m} \mathbf{s} \)

- Differs from classical charged object by factor 2

- Particle physicists say \( \mathbf{\mu} = g \frac{q}{2m} \mathbf{s} \) \( g = 1 \) classical object \( g = 2 \) Dirac particle

- \( g = 2.8 \) for proton, –1.9 for neutron \( g \) composite particles
Anomalous Magnetic Moment

- $\mu$ of electron and muon known very accurately
  - $\mu_{\text{electron}} = 2.002319304374 \pm 0.000000000008$
  - $\mu_{\text{muon}} = 2.002331832 \pm 0.0000000012$
- Not pure Dirac particles, but surrounded by thin cloud of virtual particles due to quantum fluctuation
- Measurement uses spin precession
  - Store particles with known spin orientation in B field
  - Measure spin direction after time $t$

\[ \omega \text{precess} = -\frac{e\gamma}{2m} B \]

Need to know B very accurately

Muon $g$–2 Experiment

- Muon $g$–2 Experiment
  - BNL E-821 muon storage ring
  - $\mu_{\text{muon}} = 2.0023318404 \pm 0.0000000030$

Summary

- Analyzed the motion of a heavy top
  - Reduced into 1-dimensional problem of $\theta$
  - Qualitative behavior $\Rightarrow$ Precession + nutation
  - Initial condition vs. behavior
- Magnetic dipole moment of spinning charged object
  - $M = \gamma L$, where $\gamma = q/2m$ is the gyromagnetic ratio
  - L precesses in magnetic field by $\omega = -\gamma B$
- $\gamma$ of elementary particles contains interesting physics
- Done with rigid bodies
  - Next: Oscillation