Special Relativity

- You know it quite well already
- Learned in Physics 15a/16
- We skip all the historical discussions
  - E.g. about non-existence of absolute time
- Start from two principles and build a clean formalism
  - Laws of physics are the same in all inertial frames
  - Speed of light in vacuum is the same in all inertial frames
  - Maxwell’s equations are always correct

Frames and Events

- Consider two inertial frames $S$ and $S'$
  - An event is specified by the position and the time
    $(t, x, y, z)$ in $S$ and $(t', x', y', z')$ in $S'$
  - Consider the distance between two events 1 and 2
    - We can always move the origins so that event 1 is $(t, x, y, z) = (t', x', y', z') = (0, 0, 0, 0)$
    - Physics does not depend on absolute position or time
    - We denote event 2 with $(t, x, y, z)$ and $(t', x', y', z')$
Spacetime Distance

- Light is emitted at event 1 and detected at event 2.
- Speed of light, measured in $S$ and $S'$, must be the same:
  \[ c = \frac{\sqrt{x^2 + y^2 + z^2}}{t} = \frac{\sqrt{x'^2 + y'^2 + z'^2}}{t'} \]
  \[ c^2 t^2 - (x^2 + y^2 + z^2) = 0 \]
  \[ c^2 t'^2 - (x'^2 + y'^2 + z'^2) = 0 \]
- Define distance in spacetime \( (\Delta s)^2 = c^2 t^2 - (x^2 + y^2 + z^2) \)

We assume that the spacetime distance \( \Delta s \) between any two events is the same in all inertial frames.

Light Cone

- \( (\Delta s)^2 = c^2 t^2 - (x^2 + y^2 + z^2) = 0 \) represents light:
  - If an object slower than light travels from event 1 to 2, \( t \) must be larger \( \Rightarrow (\Delta s)^2 > 0 \)
  - If \( (\Delta s)^2 < 0 \), there isn’t enough time for even light to reach

Interval between two events can fall into 3 regions:

- \((\Delta s)^2 > 0 \) \( \iff \) timelike \( \iff \) Reachable by ordinary objects
- \((\Delta s)^2 < 0 \) \( \iff \) spacelike \( \iff \) Reachable only by tachyons \((v > c)\)
- \((\Delta s)^2 = 0 \) \( \iff \) lightlike \( \iff \) Reachable by light

Light Cone

- \((\Delta s)^2 = 0 \) draws a cone in the 4-dimensional space.
- Light cone divides spacetime into past, future and elsewhere.
- Division is frame-independent:
  - Protects causality
  - Past is past, future is future, no matter which frame you are in.
**Time Dilation**

- Consider an object moving at a velocity $v$ in $S$.
  \[(dx, dy, dz) = (v_x, v_y, v_z) \text{d}t\]
- Define $S'$ so that this object is constantly at its origin.
- Consider small movement from event 1 to event 2.
  \[(\Delta s)^2 = (c\Delta t)^2 \text{ in } S'\]
  \[= (c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) \text{ in } S\]
- Make it infinitesimal.
  \[(cdt)^2 = (cdt)^2 - (dx^2 + dy^2 + dz^2) = (c^2 - v^2)dt^2\]

**4-Vectors**

- Express an event $(t, x, y, z)$ as a 4-dimensional vector.
  \[x = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad r = \begin{bmatrix} c \\ r_x \\ r_y \\ r_z \end{bmatrix}\]
- $c$ is there to fix the unit.
- Length of this vector in “ordinary” 4-d space would be.
  \[|s| = \llbracket x \rrbracket = c^2t^2 + x^2 + y^2 + z^2\]
- It’s more useful to define “length” as.
  \[|s| = x \cdot x = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = c^2t^2 - (x^2 + y^2 + z^2)\]

**Minkowski Space**

- 4-d space in which the length is defined as above is called the Minkowski space.
- Metric tensor $g$ expresses how length is defined.
- Metric tensor for ordinary space = unit matrix $I$.
- We now say.
  Length of any 4-vector in Minkowski space is the same in all inertial frames.
Lorentz Transformation

- So far we have not specified how $S$ and $S'$ are related
- The Question Of The Day is

What is the general form of the transformation $(ct, x, y, z) \rightarrow (ct', x', y', z')$ that keeps the spacetime distance $\Delta s^2$ invariant?

- We know the name: Lorentz transformation
- Let’s figure out its characteristics

What is the general form of the transformation that conserves the length in the Minkowski space?

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Linearity

- Physics does not depend on absolute position or time
  - Shifting the origin of $S$ must simply shift the origin of $S'$
  - $x \rightarrow x + a \quad \Rightarrow \quad x' \rightarrow x' + a'$
    i.e. $L(x + a) = L(x) + a$
- Try $x = 0 \Rightarrow L(a) = L(0) + a'$
  - If we define $L'(x) = L(x) - L(0)$
    $L'(x + a) = L'(x) + L'(a)$
    $\Rightarrow$ $L'$ is linear
  - $L(x)$ must be a linear function of $x + \text{fixed offset}$

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Poincaré Transformation

- Knowing that the transformation is linear
  $$
  \begin{bmatrix}
  ct' \\
  x' \\
  y' \\
  z'
  \end{bmatrix}
  =
  \begin{bmatrix}
  ct \\
  x \\
  y \\
  z
  \end{bmatrix}
  \begin{bmatrix}
  L' & 0 & 0 & 0 \\
  0 & L' & 0 & 0 \\
  0 & 0 & L' & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  $$
  Any linear transformation is written as $x' = Lx + a$.
  - This is called Poincaré transformation
  - $a$ moves the origin $\Rightarrow$ No physical consequences
- We ignore the translation and consider $L$ only
  - This is called homogeneous Lorentz transformation

Let me write this as “HLT”
Lorentz Transformation

- Now we have $x' = Lx$ → $L$ is a 4x4 real matrix
- What are the constraints on $L$?
- $L$ must conserve the length of any 4-vector
  - $|k|^2 = g_{xx}$ → $|k'|^2 = g_{xx}' = \frac{1}{L}g_{xx}L$
  - In terms of components
    - $L_{xx}g_{xx}L_{xx} = g_{xx}$ → 16 real equations
  - $g$ is symmetric → Equation is symmetric
    - 6 of 16 equations are duplicates
  - There are 10 constraints
    → $L$ has 6 degrees of freedom

Rotation

- Remember how rotation matrices were defined?
  - Conserve the length of any 3-vector in Cartesian space
  - In fact any rotation $A$ in 3-space satisfies the condition
    - $|\mathbf{k}|^2 = c^2t^2 - (x^2 + y^2 + z^2) = \text{const}$
  - 3-d rotation is a subset of HLT
    - Not the most exciting part of it
    - Rotation has 3 degrees of freedom (Euler angles)
    - There must be 3 more in HLT

Lorentz Boost

- Consider two inertial frames $S$ and $S'$
  - Origin of $S'$ must be moving at a constant velocity in $S$
    - $(x', y', z') = 0$ → $x = vt$, $y = vt$, $z = vt$
  - Rotate $S$ and $S'$ so that $x$ and $x'$ are parallel to $v$
    - Plus, $y$ and $y'$; $z$ and $z'$ are parallel to each other
    - OK to do this because rotation is a subset of HLT

\[
\begin{bmatrix}
  c \tau \\
  x' \\
  y' \\
  z'
\end{bmatrix} = \begin{bmatrix}
  L_{xx} & L_{xy} & L_{xz} & 0 \\
  L_{yx} & L_{yy} & L_{yz} & 0 \\
  L_{zx} & L_{zy} & L_{zz} & 0 \\
  0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
  c \tau \\
  x \\
  y \\
  z
\end{bmatrix}
\]

\[y \text{ and } z \text{ are out of the game}\]
Lorentz Boost

- \( \mathbf{L} \) must satisfy \( \mathbf{L}g \mathbf{L}^T = g \)
- Origin of \( S' \) is \( \mathbf{x}' = 0 \)
- This must satisfy \( x' = \gamma (v \cdot x + ct') \)
- Looks familiar except…

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}
\]

Sign Ambiguities

- There are 4-fold sign ambiguities
- Only 1 represents Lorentz transformation

- Think about the low-velocity limit

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}
\]

Space-Time Reversal

- What are the other solutions?
- They reverse the orientation of time and/or space

- Flips \( x \) axis: Space inversion
- Flips \( t \) : Time reversal

- Laws of physics are happy with such transformation
- Not “continuous” part of the Lorentz transformation
- We aren’t interested for the moment
General Boost

- We’ve got \( \mathbf{L} \) for boost in \( x \)
  - We can rotate \( S \) and \( S' \) to get \( \mathbf{L} \) for boost in any direction
- We can also use a bit of vector algebra
  - Split 3-vector \( \mathbf{r} \) into two parts
    - Parallel component transforms like \( x \) above
      \[
      c't' = \gamma ct - \beta \beta |\mathbf{v}| = \gamma (ct - \beta \mathbf{v} \cdot \mathbf{r})
      \]
      \[
      \mathbf{r}' = -\gamma \beta ct + \gamma \mathbf{r} + r_i = -\gamma \beta ct + \mathbf{r} + \frac{(\mathbf{r} \cdot \mathbf{v})(\gamma - 1)}{\beta^2}
      \]
  - Parallel component transforms like \( x \) above

Proper Lorentz Transformation

- Writing down explicitly for general \( \beta \)
  \[
  \mathbf{L} = \begin{bmatrix}
  \gamma & -\beta \beta & -\gamma \beta_i \\
  -\beta \beta & 1 + (\gamma - 1) \frac{\beta \beta}{\beta^2} & (\gamma - 1) \frac{\beta \beta}{\beta^2} \\
  -\gamma \beta_i & (\gamma - 1) \frac{\beta \beta}{\beta^2} & 1 + (\gamma - 1) \frac{\beta \beta}{\beta^2}
  \end{bmatrix}
  \]
  \[3 \text{ degrees of freedom } (\beta_x, \beta_y, \beta_z)\]
  \[\Rightarrow \text{ General form of Lorentz transformation without rotation}\]

Lorentz T at a Glance

- Proper Lorentz Transformation
  - 3 degrees of freedom in \( \beta \)
  - Orientation of the axes are unchanged
- Homogeneous Lorentz Transformation \( \mathbf{L} \mathbf{A} \)
  - \( \mathbf{A} \) is rotation, \( 3 + 3 = 6 \) degrees of freedom
  - Origin is unchanged
- Poincaré transformation \( \mathbf{X} = \mathbf{L} \mathbf{A} + \mathbf{a} \)
  - \( \mathbf{a} \) shifts the origin, \( 6 + 4 = 10 \) degrees of freedom
  - Completely general form of transformation between inertial systems that satisfy special relativity
Properties of $L$

- $L \rightarrow 1$ when $\beta \rightarrow 0$
- $L^{-1}$ is given by $\beta \rightarrow -\beta$

$$
\begin{pmatrix}
\gamma & -\gamma ' \\
-\gamma ' & \gamma \\
1 + (\gamma - 1) \frac{\beta \beta'}{\beta^2} & 1 + (\gamma - 1) \frac{\beta \beta'}{\beta^2}
\end{pmatrix} = 1
$$

- Diagonally symmetric
  - Remember: rotation matrix $A$ is anti-symmetric
  - $LA$ (for HLT) is neither symmetric nor anti-symmetric

Addition of Velocities

- Make two PLTs $L$ and $L'$ in series
  - If $\beta$ and $\beta'$ are parallel, we can define $x$ as their direction

$$
L/LL' =
\begin{pmatrix}
\gamma & -\gamma ' \\
-\gamma ' & \gamma \\
\gamma (1 + \beta \beta') & -\gamma (\beta + \beta') \\
-\gamma (\beta + \beta') & \gamma (1 + \beta \beta')
\end{pmatrix}
$$

- Same as one PLT with $\beta'' = \frac{\beta + \beta'}{1 + \beta' \beta}$

- $|\beta''| < 1$ for any $\beta$ and $\beta'$ as expected

Addition of Velocities

- What if we add two velocities that are not parallel?
  - For example, $\beta$ is in $x$, $\beta'$ is in $y$

$$
L/LL' =
\begin{pmatrix}
\gamma & 0 & -\gamma ' \\
0 & 1 & 0 \\
-\gamma ' & 0 & \gamma \\
\gamma (1 + \beta \beta') & -\gamma (\beta + \beta') & \gamma (1 + \beta \beta')
\end{pmatrix}
$$

- Obviously $L/LL' \neq LL'$

- PLTs are not commutative
### Addition of Velocities

- In general, $L'L$ is asymmetric
  - Two PLTs do not add up to a PLT
- On the other hand, two HLTs do add up to a HLT
  - HLTs are defined as “linear transformations that conserve length of 4-vectors and do not move the origin”
    - Two successive HLTs automatically satisfy this
- Therefore two PLTs must make a HLT
  - i.e., $L'L$ must be written as $L'A$
  - i.e., PLT + PLT = PLT + a rotation
  - Goldstein section 7.3 makes a big fuss about this

### Lorentz Group

- Homogeneous Lorentz transformations make a closed group (Lorentz group)
  - Product of two members is always a member
- Rotations make a group
  - Which is a subgroup of the Lorentz group
  - Proof: Euler’s theorem
- Proper Lorentz transformations do not make a group

### Summary

- Discussed Lorentz transformation
  - Linear transformation of 4-vectors that conserve the length in Minkowski space
- Derived general form of homogeneous Lorentz transformation
  - Product of rotation and proper Lorentz transformation
  - Found explicit matrix expression of PLT
  - HLTs form a group, PLTs don’t
- Now we know how the spacetime transforms
  - Next: physical quantities in the spacetime