Lecture 15
Special Relativity
(Chapter 7)

What We Did Last Time
- Defined Lorentz transformation
  - Linear transformation of 4-vectors that conserve the length in Minkowski space
- Derived general form of homogeneous Lorentz transformation
  - Product of rotation and proper Lorentz transformation
  - Found explicit matrix expression of PLT
  - HLTs form a group, PLTs don’t
    - $\text{HLT} \times \text{HLT} = \text{HLT}$
    - $\text{PLT} \times \text{PLT} = \text{HLT}$

Fun With Paradox
- In general, two PLTs don’t add up to a PLT
  - Rotation becomes involved
- Example: two objects are moving in parallel
  - Can you see where the rotation come from?
Fun With Paradox

- How do you know two objects passed the line “simultaneously”?
  - By sending light and receiving reflection

Light is sent at $t = \frac{l}{c}$
Reflections come back at $t = \frac{l}{c}$

Took same time ↭ same distance
Came back simultaneously ↭ reflected at the same time

- What happens if the observer was moving?

Fun With Paradox

- For a moving observer

Light is sent at the same moment
Reflection from A comes back earlier than from B
⇒ A must have passed earlier!

Definition of “simultaneous” depends on the observer
- Causing this effect

What We Did Last Time

- Discussed Lorentz transformation
  - Linear transformation of 4-vectors that conserve the length in Minkowski space

- Derived general form of homogeneous Lorentz transformation
  - Product of rotation and proper Lorentz transformation
  - Found explicit matrix expression of PLT
  - HLTs form a group, PLTs don’t

- We got the stage = spacetime
  - Let’s get the actors = physical quantities
4-Vectors

- We write 4-vectors as $x^{\mu}$.
  - Greek index = $0 \ldots 3$
  - $x^{0} = ct$  
  - $x^{1} = x$  
  - $x^{2} = y$  
  - $x^{3} = z$
- It seems confusing, but you’ll get used to it.
- Let’s follow a particle traveling in 4-space.
  - Trajectory is given by $x^{\mu}(\lambda)$, $x^{\lambda}(\lambda)$.
  - $\lambda$ is a parameter that varies monotonously along the curve.
  - Proper time $\tau$ is a convenient possibility for $\lambda$.
- At any point on the curve, we can define a tangent 4-vector.

\[ \mu^{\tau} = \frac{d \lambda}{d \tau} \]

Call it 4-velocity.

4-Velocity

- $S$ is observer’s frame. $S'$ is the particle’s rest frame.
- Particle’s 3-velocity in $S$ is $v$.
  - $x^{0} = ct = \gamma vt$  
  - $x^{1} = x = \gamma v' ct = \gamma v' t$
- We define 4-velocity.

\[ u^{\mu} = \frac{\partial x^{\mu}}{\partial \tau} = \gamma v^{\mu} \]

- “Length” is $u^{\mu}u_{\mu} - u'^{\mu}u'^{\mu} = \gamma v^{\mu}v_{\mu} = c^{2}$.
- 4-velocity is the relativistic extension of $v$.

4-Momentum

- Multiplying 4-velocity with mass gives 4-momentum.
  - $p^{0} = mu^{0} = myc$  
  - $p^{i} = mu^{i} = myv^{i}$
- Space part is natural extension of 3-momentum $p = myv$.
- Time part is energy $E = c^{2} = myc$.
  - This needs to be confirmed after introducing force.
- Lorentz invariance is obvious.
  - $p^{i}p_{i} - p'^{i}p'^{i} = m^{2}c^{2}$
- Kinetic energy is defined as

\[ T = E - E_{mc} = \sqrt{m^{2}c^{4} + p^{2}c^{2}} - mc^{2} \]
Lorentz Tensor

- Proper Lorentz transformation turns a 4-vector into another 4-vector
  - Consider it a linear function of 4-vector
  - Express it as $L^\mu_\nu x^\nu = x'^\mu$
    - Upper index = 4-vector
    - Lower index = function that accepts 4-vector
  - You can define a whole bunch of quantities using this convention ➔ Call them **tensors**
    - $X, X^\mu, X_{\mu}, X_{\mu\nu}, X^{\mu\nu}$
  - We’ll find their physical meanings as we go

Tensor Product

- Tensor product of two 4-vectors is defined by $T^{\alpha\beta} \equiv x^\alpha y^\beta$
  - Write this $T \equiv \otimes$
  - $T^{\alpha\beta}$ is a tensor of rank 2
  - You can repeat this to define tensors of rank $n$
  - Lorentz transformation of $T^{\alpha\beta}$ can be easily found
    - $T'^{\alpha\beta} = u^\alpha v^\beta = L^\mu_\alpha L^\nu_\beta g_{\mu\nu} = L^\mu_\alpha L^\nu_\beta T^{\alpha\beta}$
  - Use as many Lorentz tensor as necessary to convert all indices

Scalar Product

- We define the scalar product of two 4-vectors $u \cdot v = u^\mu v^\mu$
  - Two lower indices because it takes two 4-vectors and returns a scalar
  - How does it transform?
    - $u' \cdot v' = u'^\mu g_{\mu\nu} v'^\nu = u^\mu L^\mu_\alpha L^\nu_\beta g_{\mu\nu} = u^\mu v^\mu = u \cdot v$
  - Scalar product is Lorentz invariant, as expected
Metric Tensor

- A coordinate system in general has basis vectors
  \[ u = u^i e_i = u^i e_e + u^i e_e + u^i e_3 + u^i e_4 \]
- Scalar product \( u \cdot v \) can be written as
  \[ u \cdot v = u^i e_i \cdot v^k e_k \]
  \[ g_{ij} = e_i \cdot e_j \]

Metric tensor defines the scalar products of the basis vectors
- Lengths of and angles between the basis vectors
- That’s what “metric” means

General Metric Tensor

- Metric \( g_{\mu\nu} \) in Minkowski space is diagonal
  - Coordinate axes are always orthogonal
- Formalism of tensors allows more flexibility
  - Useful in curved space coordinates
  - General relativity makes full use of this
  - Let’s not get into it for now…

1-Form

- Let’s look at a scalar product in a different way
  \[ u \cdot v = u^\nu g_{\nu\mu} \]
  \( u \Rightarrow \text{4-vector} \)

- We call \( u_i = u^\nu g_{\nu i} \) as the 1-form of \( u^\nu \)
  \( x_i = x^i = ct \)
  \( x_i = -x^i \)

- Difference between 4-vector and 1-form seems small
  - Why do we make such distinction?
  - non-Minkowski metric can make them really different
  - There are physical quantities that are naturally 1-form
Gradient

Consider a scalar function \( f(x^\mu) \)
- A particle goes along a curve \( x^\mu = x^\mu(t) \)
- Rate of increase of \( f(x^\mu(t)) \) is given by
  \[
  \frac{df}{dt} = \frac{df}{dx^\mu} x^\mu \partial_{\mu}
  \]
- Gradient operates on velocity to make a scalar \( \rightarrow 1\)-form
- Gradient operator is defined by \( \partial_\mu \)
  - Also known as \( \text{d} \)
  - But I’ll avoid this notation
  - Lower index shows it’s a 1-form

4-Vector and 1-Form

- 4-vector can be turned into its 1-form by \( u^\mu = u^\mu g_{\mu \nu} \)
  - Obviously you can do the reverse \( u^\mu = g^{\alpha \beta} u_\alpha \)
    where \( g^{\alpha \beta} g_{\beta \gamma} = \delta^\alpha_\gamma \)
  - \( g^{\alpha \beta} \) looks identical to \( g_{\mu \nu} \) in Minkowski space
- This gives us Lorentz transformation for 1-form
  \[
  n^\nu = g^{\alpha \beta} n_\alpha L_{\alpha \beta} u^\beta = g^{\alpha \beta} n_\alpha L_{\alpha \beta} u^\beta = L_{\alpha} u^\alpha \]
  - Works just the same as 4-vector, as it should

Rank of Tensors

- A general tensor has \( n \) upper and \( p \) lower indices
  - Call it a tensor of rank \( \left( \begin{array}{c} n \\ p \end{array} \right) \)
  - It takes \( p \) 4-vectors and \( n \) 1-forms and return a scalar
    \( T^\alpha a_b c = \) scalar
  - 4-vector and its 1-form are interchangeable using \( g_{\mu \nu} \)
    - We can turn a tensor into equivalent tensors with different rank, as long as \( n + p \) is conserved
    - Example:
      \[
      T^\alpha a_b c = T^\alpha (g_{\alpha \beta}) b c \Rightarrow T^\alpha = T^\alpha g_{\alpha \beta}
      \]
Lorentz Transformation

- We can find Lorentz transformation for any tensor
  - Transform all indices using Lorentz tensor
  - Example:
    \[ T_\alpha^\nu a_\beta b_\mu = \text{scalar} \]
    Transform this to get
    \[ T_\alpha^\nu a_\beta b_\mu = T_\alpha^\nu L_\alpha^\gamma L_\beta^\delta L_\mu^\rho L_\nu^\sigma \]
  - We now know all the rules for
    - Lorentz transformation
    - Moving indices up and down
    - They aren’t even all that complicated

Force

- Newton’s laws must be correct if the velocity is zero
  - \[ F = \frac{dp}{dt} \] in the rest frame of the object
  - Momentum transforms as a 4-vector
  - Time dilation changes the time derivative
  - Natural extension would be
    \[ \frac{dp}{d\tau} = K^\nu \]
    - \( K^\nu \) must be a 4-vector
    - \( \tau \) is proper time. Connected with \( t \) by \( dt = \gamma d\tau \)
  - How do we find the 4-force \( K^\nu \)?

Electromagnetic Force

- We assume Maxwell’s equations are always correct
  - Predicts constant speed of light \( c \)
  - We want to rewrite it in a covariant form
  - EM force on a charged particle can be derived from the generalized potential \( U = e(\phi - A \cdot v) \)
  - 4-velocity \( u_\mu \) was \( (u_\mu, u) = (v_x, v_y) \)
  - Define 4-potential as \( A = (A, A) = (\phi/c, A) \)
  - Scalar product is
    \[ A^\mu u_\mu = \frac{\phi}{c} v_x - A_y v_y = \gamma (\phi - A \cdot v) \]
    - New \( \gamma U = eA^\mu u_\mu \) looks promising
Electromagnetic Force

- $U = \epsilon(A' u_\mu)$ is a scalar if $A' = (\phi, cA)$ is a 4-vector
  - What we are looking for if the laws of EM are covariant
- Force in 3-d is given by $F_i = -\frac{\partial U}{\partial x_i} + \frac{d}{dt} \left( \frac{\partial U}{\partial v_i} \right)$
  - Extend this to 4-d
    - Careful to make it a real 4-vector
      
      $$K^\nu = \frac{\partial e(A' u_\mu)}{\partial x_\mu} \frac{dA^\nu}{dt} = \left( \frac{\partial A^\nu}{\partial x_\mu} u_\mu - \frac{\partial A^\nu}{\partial x_\tau} u_\tau \right)$$

  Minkowski force for a charged particle in EM field

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Electromagnetic Force

- $K^\nu = \epsilon \left( \frac{\partial A^\nu}{\partial x_\mu} u_\mu - \frac{\partial A^\nu}{\partial x_\tau} u_\tau \right)$
  - Rewrite using $E$ and $B$
    
    $$E' = -\nabla \phi - \frac{\partial A^\mu}{\partial t} = \left( \frac{\partial A^\nu}{\partial x_\mu} \frac{\partial A^\mu}{\partial x_\nu} \right)$$
    
    $$\{ (v \times B) \} = (v \times (\nabla \times A)) = \left( \frac{\partial A^\nu}{\partial x_\mu} \frac{\partial A^\mu}{\partial x_\nu} \right)$$

    A bit of work

    $$K^\nu = \frac{\gamma}{c} e' E'$$
    $$K^\nu = \gamma e \left[ E' + (v \times B) \right]$$

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Electromagnetic Force

- Space part
  
  $$\frac{dp^\mu}{d\tau} = K^\nu = \gamma e \left[ E' - (v \times B) \right]$$
  
  - Agrees with
    
    $$\frac{dp}{dt} = F = e \left[ E - (v \times B) \right]$$

- Time part
  
  $$\frac{dp^\mu}{d\tau} = K^\nu = \frac{\gamma}{c} e' E$$

  - Rate of work done by the EM force
    
    $$\frac{dW}{dt} = \frac{E}{c}$$

  - Energy!

  - Confirmation promised earlier
Faraday Tensor

\[ F^{\mu \nu} = e \left( \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \right) = \left( \begin{array}{ccc} 0 & -E_z & -E_y \\ -E_z & 0 & -cB_z \\ -E_y & cB_z & 0 \end{array} \right) \]

- Tensor \( F^{\mu \nu} \) is called Faraday, or EM field tensor
- \( E \) and \( cB \) form a tensor of rank 2
- cf. \( \Phi \) and \( A \) make a 4-vector

Other Forces

- What happens with forces other than EM?
- There is no general method for making forces covariant
- You must deal with each force, case by case
- There are 4 (known) fundamental forces in nature
  - EM, gravity, weak and strong
- Covariant form has been found for weak and strong
  - Need quantum field theory to do this
- Gravity cannot be made covariant
  - Need general relativity

Summary

- Defined covariant form of physical quantities
  - 4-vectors: velocity, momentum
  - Tensors: metric, Lorentz
  - 1-forms: gradient
- Found how to Lorentz transform them
- Covariant form of Newton’s equation with EM force
  - EM potential \( \Phi \) \rightarrow 4-vector, EM force \( A \) \rightarrow tensor
  - Equation of motion \( \frac{d\Phi}{dt} = K^\mu \)