Lecture 16
Special Relativity
(Chapter 7)
What We Did Last Time

- Defined covariant form of physical quantities
  - Collectively called “tensors”
    - Scalars, 4-vectors, 1-forms, rank-2 tensors, …
  - Found how to Lorentz transform them
    - Use Lorentz tensor & metric tensor
- Covariant form of Newton’s equation with EM force
  - Equation of motion $\frac{dp^\mu}{d\tau} = K^\mu$
  - EM potential $\to$ 4-vector $(\phi/c, A)$
  - EM field $\to$ Faraday tensor $\leftarrow$ I gave you a wrong one…
Faraday Tensor

\[ K^\mu = e \left( \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) u_\nu \equiv e F^{\mu\nu} u_\nu \]

- \( F^{\mu\nu} \) is derivatives of the EM potential \( \rightarrow \) \( E \) and \( B \) fields

\[ E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} = c \left( \frac{\partial A^0}{\partial x_1} - \frac{\partial A^1}{\partial x_0} \right) \]

\[ B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\partial A^3}{\partial x_2} + \frac{\partial A^2}{\partial x_3} \]

\[ F^{\mu\nu} = \begin{bmatrix}
0 & -E_x/c & -E_y/c & -E_z/c \\
E_x/c & 0 & -B_z & B_y \\
E_y/c & B_z & 0 & -B_x \\
E_z/c & -B_y & B_x & 0
\end{bmatrix} \]

What I gave you (and the textbook) was wrong by a factor \( c \)
Multi-Particle System

- Consider a system with particles \( s = 1, 2, \ldots \)
  - Total momentum \( P^\mu = \sum_s p_s^\mu \)
  - Equation of motion for each particle \( \frac{dp_s^\mu}{d\tau_s} = K_s^\mu \)
    - Different time for each particle!

- EoM for the total momentum is
  - Not a very clean equation

\[
\frac{dP^\mu}{dt} = \sum_s \gamma_s \frac{dp_s^\mu}{d\tau_s} = \sum_s \gamma_s K_s^\mu
\]

Trouble ahead…
Momentum Conservation

- Imagine a 2-particle system with no external force
- But there are internal forces between the particles
- Law of action and reaction $K_{1\rightarrow 2} = -K_{2\rightarrow 1}$

$$\frac{dP}{dt} = \sum_s \gamma_s \frac{dp_s}{d\tau_s} = \sum_s \gamma_s K_s$$

$$\frac{dP_{1+2}}{dt} = \gamma_1 K_{2\rightarrow 1} + \gamma_2 K_{1\rightarrow 2}$$

- To conserve the total momentum in all frames, the particles interacting with each other must have the same velocity

This is zero only if $\gamma_1 = \gamma_2$

Is this a weird restriction, or what?
Local Interaction

- Suppose the particles interact only when in contact
  - Forces exchanged only when they collide
  - They share the rest frame momentarily → Same $\gamma$

Total momentum of a multi-particle system is conserved if the interactions between the particles are local

- Furthermore, Law of action/reaction $K_{1\rightarrow 2} = -K_{2\rightarrow 1}$ cannot work over a distance instantaneously
  - There must be delays → Conservation laws cannot hold

Relativity and non-local interactions don’t mix
Particle Collisions

- Interactions between particles must be local
  - Force exchanged when they collide
  - Free motion between collisions
- Consider the collision as a black box
  - We don’t know what happens in the box (not classical)
  - Motion outside the box is easy → Relativistic Kinematics
- How much can we learn without opening the box?
Center-of-Momentum Frame

- Local interactions conserve total 4-momentum
  - i.e., total energy and total 3-momentum are conserved

\[
p^\mu = \sum_{s=1}^{n} p_s^\mu = \left( \frac{E}{c}, \mathbf{p} \right) \quad \leftrightarrow \quad E = \sum_{s=1}^{n} E_s \quad \mathbf{p} = \sum_{s=1}^{n} \mathbf{p}_s
\]

- We know how to Lorentz transform it

\[
p'^\mu = \sum_s p'^\mu_s = \sum_s L^\mu_\nu p^\nu_s = L^\mu_\nu p^\nu
\]
as usual

- Define the center-of-momentum frame in which \( \mathbf{p} = 0 \)
  - It’s the frame in which the total 3-momentum is zero
  - Or, the center of mass is at rest
  - Often called center-of-mass frame as well
CoM Energy and Boost

There are two particularly useful quantities

- **CoM energy** $E'$
  - Lorentz invariance $p_\mu p_\mu = p'_\mu p'_\mu$
  - $E'$ is the smallest possible $E$

- **Boost $\beta$ of CoM frame relative to the lab. frame**
  - Lorentz transformation
    

$$p^\mu = \sum_{s=1}^{n} p_s^\mu = \left( \frac{E}{c}, \mathbf{p} \right) \quad \text{CoM frame} \quad p'^\mu = \left( \frac{E'}{c}, 0 \right)$$

$$\begin{align*}
E'^2 &= \frac{E^2}{c^2} - \mathbf{p}^2 \\
\gamma &= \frac{E}{E'} \\
\beta &= \frac{\mathbf{p}c}{E}
\end{align*}$$
Two-Particle Collision

Consider collision of particle 1 on particle 2 at rest

\[ p_1 = \left( \frac{E_1}{c}, \mathbf{p}_1 \right) \quad p_2 = (m_2 c, \mathbf{0}) \]

- Total 4-momentum is \( p = (E_1/c + m_2 c, \mathbf{p}_1) \)
- Total CoM energy
  \[ E'^2 = p^\mu p_\mu c^2 = (m_1^2 + m_2^2) c^4 + 2E_1 m_2 c^2 \]
- Boost of CoM frame is
  \[ \beta = \frac{\mathbf{p}_1}{E_1/c + m_2 c} = \frac{m_1 \gamma_1 \mathbf{v}_1}{m_1 \gamma_1 c + m_2 c} \]

Fixed-target collision

\( E' \) grows slowly with \( E_1 \)

Approaches \( v_1/c \) for large \( E_1 \)
Creation Threshold

- Suppose we are trying to create a new particle
  - In the best scenario, particles 1 and 2 merge to create a new heavy particle 3
  - Total 4-momentum would be simply $p = p_1 + p_2 = p_3$
  - Total CoM energy is $E' = m_3 c^2$
  - How much energy $E_1$ do we have to give particle 1?
    $$ E'^2/c^2 = p^\mu_3 p_{3\mu} = m_3^2 c^2 \quad \Rightarrow \quad E' = m_3 c^2 $$
    $E' = (m_1^2 + m_2^2) c^4 + 2 E_1 m_2 c^2 = m_3 c^4$ 
    $E_1 = \frac{(m_3^2 - m_1^2 - m_2^2) c^2}{2m_2}$
  - For large $m_3$, $E_1$ grows with $m_3^2$
Fixed-Target vs. Collider

- Consider hitting a proton with a proton to make a Higgs particle, which is $X$ times heavier than a proton
  \[ E_1 = \frac{(m_3^2 - m_1^2 - m_2^2)c^2}{2m_2} \approx \frac{X^2}{2} m_p c^2 \]

- For $X > 100$, we’d need a >5000 GeV accelerator

- Particle colliders are more energy-efficient

- Laboratory is CoM
  \[ p = (E_1 + E_2, 0) \]

- Just need
  \[ E_1 + E_2 = m_3 c^2 = X m_p c^2 \]

  $50 \text{ GeV} + 50 \text{ GeV}$
Elastic Scattering

- Particle 1 hits particle 2 and get elastically scattered

- Cross section is calculated in CoM frame
  - By treating it as a central-force problem
- Experiment is done in the laboratory frame
- We need to learn how to translate between the CoM and the laboratory frames
Elastic Scattering

First, what’s the boost?

- Total momentum is
  \[ p^\mu = p_1^\mu + p_2^\mu = \left( \frac{E_1}{c} + m_2c, p_1 \right) \]

- \[ \beta = \frac{p^0}{p^0} = \frac{p_1 c}{E_1 + m_2c^2} \]

- Let’s get \( \gamma \) as well

  \[ p^\mu p_\mu = \left( \frac{E_1}{c} + m_2c \right)^2 - p_1^2 = m_1^2c^2 + m_2^2c^2 + 2E_1m_2 \]

  \[ \gamma = \frac{p^0}{\sqrt{p^\mu p_\mu}} = \frac{E_1 + m_2c^2}{\sqrt{m_1^2c^4 + m_2^2c^4 + 2E_1m_2c^2}} \]
Elastic Scattering

- Now we can boost $p_1$ to CoM

$$p_1^{\mu} = (E_1/c, p_1, 0, 0) \Rightarrow p_1^{\prime\mu} = (E_1'/c, p_1', 0, 0)$$

- $p_3'$ is given by rotating $p_1'$ by $\Theta$

$$p_3'^{\mu} = (E_1'/c, p_1' \cos \Theta, p_1' \sin \Theta, 0)$$

- Boost this back to get $p_3$

$$E_3 = \gamma(E_1' + \beta p_1' c \cos \Theta) \quad p_3^1 = \gamma(p_1' \cos \Theta + \beta E_1'/c) \quad p_3^2 = p_1' \sin \Theta$$

- Scattering angle in lab frame is

$$\tan \vartheta = \frac{p_3^2}{p_3^1} = \frac{\sin \Theta}{\gamma(\cos \Theta + \beta E_1'/(p_1'c))} = \frac{\sin \Theta}{\gamma(\cos \Theta + \beta / \beta_1')}$$

Velocity of 1 in CoM
Elastic Scattering

- What happens to the kinetic energy?

\[ E_3 = \gamma^2 \left[ (1 - \beta^2 \cos \Theta)E_1 - \beta (1 - \cos \Theta)p_1 c \right] \]

- At \( \Theta = 0 \)

\[ E_3 = \gamma^2 (1 - \beta^2)E_1 = E_1 \]  

Makes sense

- With a little bit of work

\[ \frac{T_3}{T_1} = 1 - \frac{2\rho (1 + \epsilon_1 / 2)}{(1 + \rho)^2 + 2\rho \epsilon_1} (1 - \cos \Theta) \]

Sign wrong in textbook

- Worst case is \( \Theta = \pi \)

\[ \frac{(T_3)_{\text{min}}}{T_1} = \frac{(1 - \rho)^2}{(1 + \rho)^2 + 2\rho \epsilon_1} \]

\[ \epsilon_1 = \frac{T_1}{m_1 c^2} \]
\[ \rho = \frac{m_1}{m_2} \]
Elastic Scattering

\[ \frac{(T_3)_{\text{min}}}{T_1} = \frac{(1 - \rho)^2}{(1 + \rho)^2} \]

- **Non-relativistic limit**
  
  If \( m_1 \ll m_2 \), i.e., the target is heavy, almost no energy is lost in the collision

- **Ultra-relativistic limit**

  \[ \frac{(T_3)_{\text{min}}}{T_1} = \frac{(1 - \rho)^2}{2 \rho \mathcal{E}_1} = \frac{(m_2 - m_1)^2 c^2}{2m_2 T_1} \]

  \((T_3)_{\text{min}}\) is independent of \( T_1 \)

- As \( T_1 \) increases, the energy loss becomes very large
Particle Decays

- Some particles are unstable and decay after a while

- Mass of the “mother” is known from the 4-momenta of the “daughters” by calculating the total CoM energy

- This is how particle physicists find particles that do not live long enough to be directly seen

- Example: Discovery of $J/\psi$ (November 1974)
Particle Decays

- A day at the BABAR experiment at SLAC
  - Collide $e^+$ and $e^-$ to generate a few 100,000 $\Upsilon(4S)$ particles
  - … each of which decays into two $B^0$ mesons
  - … some of which decays into a $J/\psi$ and a $K_S$
  - … $J/\psi$ decays into $e^+$ and $e^-$, or $\mu^+$ and $\mu^-$
  - … $K_S$ decays into $\pi^+$ and $\pi^-$
  - Measure 3-momenta of the stable particles
    - Masses known $\rightarrow$ Calculate 4-momenta
  - Rebuild the decay chain backwards and calculate invariant masses of them all $\rightarrow$ Do they match the expected masses?
J/ψ mass

Combine $e^+e^-$ or $\mu^+\mu^-$ and to see if they make a J/ψ.
Combine $J/\psi$ and $K_S$ to make $B^0$

**BABAR**

Found 440 signals out of 30 million $\Upsilon(4S)$ generated
Summary

- Discussed multi-particle system
  - Only local forces are amiable with relativity
- Relativistic kinematics for particle physics
  - CoM frame, CoM energy, invariant mass and boost
  - Particle creation and collider experiment
  - Elastic scattering
  - Particle decays
- Next lecture: Lagrangian formalism