Lecture 17
Special Relativity
(Chapter 7)
What We Did Last Time

- Worked on relativistic kinematics
  - Essential tool for experimental physics
  - Basic techniques are easy:
    - Define all 4 vectors
    - Calculate c-o-m energy and boost
    - Go about with business
  - Examples:
    - Particle creation
    - Elastic scattering
    - Particle decays
Today’s Goals

- Relativistic Lagrangian formulation
  - Two different approaches: practical and truly relativistic
  - Neither is perfect – Will cover both
    - Will do a few easy examples in the process
Lagrangian Formulation

- **Proper Approach**
  - Set up a covariant form of Hamilton’s principle
  - Keep everything in clean tensor forms

- **Practical Approach**
  - Build a Lagrangian that reproduces 3-force in a frame
  - May or may not be correct in other frames
  - Works OK pretty often, but no guarantee
For a single particle of mass $m$

$$L = -m c^2 \sqrt{1 - \beta^2} - V(x)$$

$\beta = \text{reduced velocity}$

Let’s check if this works

$$\frac{\partial L}{\partial v^i} = -mc \frac{\partial \sqrt{1 - \beta^2}}{\partial \beta^i} = \frac{mc \beta^i}{\sqrt{1 - \beta^2}} = p^i$$

Space component good. But no time component

3-d equation of motion

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial x^i} = \dot{p}^i + \frac{\partial V}{\partial x^i} = \dot{p}^i - F^i = 0$$

Looks OK for the 3-d part… Try to push this path
Expand the definition to allow $v$-dependent potential

Consider the EM force

$$L = -mc^2 \sqrt{1 - \beta^2} - U(x, v) = -mc^2 \sqrt{1 - \beta^2} - q\phi + qA \cdot v$$

We know that $U$ gives us

$$-\frac{\partial U}{\partial x^i} + \frac{d}{dt} \left( \frac{\partial U}{\partial v^i} \right) = qE^i + q(v \times B)^i$$

Only difference is the definition of the momentum

$$\mathcal{P}^i = \frac{\partial L}{\partial \dot{v}^i} = p^i + qA^i$$

Canonical momentum

Classical 3-momentum

Did this before

Still works fine

Same thing happened without relativity

No big deal
Energy Function

Energy function $h$ is defined by

$$h = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L = mc^2 \beta^i v^i \sqrt{1 - \beta^2} + mc^2 \sqrt{1 - \beta^2} + V = \frac{mc^2}{\sqrt{1 - \beta^2}} + V$$

- This is total energy
- It’s conserved if $V$ is time-independent
- Proved this before – No changes by going relativistic
Simple Example

- Particle accelerating under constant force
  - Electron in an electric field
    \[ L = -mc^2 \sqrt{1-\beta^2} + eEx \]
  - Lagrange’s equation
    \[
    \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}} \right) - \frac{\partial L}{\partial x} = \frac{d}{dt} \left( \frac{mc\beta}{\sqrt{1-\beta^2}} \right) - eE = 0
    \]
    \[
    \frac{d}{dt} \left( \frac{\beta}{\sqrt{1-\beta^2}} \right) = \frac{eE}{mc}
    \]
  - Integrate twice, assuming \( x = 0, \dot{v} = 0 \) at \( t = 0 \)
    \[
    \beta = \frac{eE}{mc} t \sqrt{1 + \left( \frac{eE}{mc} t \right)^2}
    \]
    \[
    x = \frac{mc^2}{eE} \left( \sqrt{1 + \left( \frac{eE}{mc} t \right)^2} - 1 \right)
    \]
Simple Example

Relativistic solution is a hyperbola
- Approaches $v = c$
- Non-relativistic solution (parabola) accelerates faster

\[ x = \frac{mc^2}{eE} \left( \sqrt{1 + \left( \frac{eE}{mc} t \right)^2} - 1 \right) \]
Simple Example

\[ \beta = \frac{eE/mc}{1 + (eE/mc)^2 t} \]

\[ x = \frac{mc^2}{eE} \left( \sqrt{1 + \left(\frac{eE}{mc} t\right)^2} - 1 \right) \]

- Low-velocity limit \( \rightarrow \)  
  \[ v = \frac{eE}{m} t \]
  \[ x = \frac{eE}{2m} t^2 \]

- \( t \rightarrow \infty \) limit \( \rightarrow \)  
  \[ \beta \rightarrow 1 \]
  \[ x \rightarrow ct \]

- Look at it in terms of energy

\[ x = \frac{mc^2}{eE} (\gamma - 1) \]

\[ eEx = mc^2 (\gamma - 1) \]

All as expected

LHS = \(-V(x)\)  \[ \text{RHS} = p^0 c - mc^2 = T \]

Energy conservation
Consider a 1-dim. harmonic oscillator

\[ L = -mc^2 \sqrt{1 - \beta^2} - V \]

\[ V = \frac{1}{2} kx^2 \]

Let’s use energy conservation this time

\[ E = \frac{mc^2}{\sqrt{1 - \beta^2}} + V = \text{const} \]

Solution exists only when

\[ E - V > mc^2 \]

Oscillation between two points expected

\[ \beta^2 = 1 - \frac{m^2 c^4}{(E - V)^2} > 0 \]

What’s the frequency?
Semi-Relativistic Oscillator

- Integrate $\beta$ for $\frac{1}{4}$ of the cycle

$$\beta = \frac{1}{c} \frac{dx}{dt} = \sqrt{1 - \frac{m^2c^4}{(E - V)^2}}$$

$$\frac{\tau}{4} = \int_0^b \frac{1}{c \sqrt{1 - \frac{m^2c^4}{(E - V)^2}}} \, dx$$

- $b$ is given by $E = mc^2 + \frac{1}{2} kb^2$

$$\frac{E - V}{mc^2} = 1 + \frac{k}{2mc^2} (b^2 - x^2) \equiv 1 + \kappa (b^2 - x^2)$$

- Approximate for $V \ll mc^2$

$$\frac{E - V}{mc^2} = 1 + \varepsilon$$

$$\frac{1}{\sqrt{1 - (1 + \varepsilon)^{-2}}} \approx \frac{1}{\sqrt{2\varepsilon - 3\varepsilon^2}} \approx \frac{1 + \frac{3}{4} \varepsilon}{\sqrt{2\varepsilon}}$$
Semi-Relativistic Oscillator

\[ \tau = \frac{4}{c} \int_{0}^{b} \frac{1 + \frac{3}{4} \kappa (b^2 - x^2)}{\sqrt{2 \kappa (b^2 - x^2)}} \, dx = \frac{2\pi}{c \sqrt{2 \kappa}} \left( 1 + \frac{3}{8} \kappa b^2 \right) = 2\pi \sqrt{\frac{m}{k}} \left( 1 + \frac{3kb^2}{16mc^2} \right) \]

- Period is longer than non-relativistic oscillator

\[ \Delta \tau \frac{3kb^2}{\tau_0 16mc^2} = \frac{3 V_{\text{max}}}{8 mc^2} \]

Wrong sign in textbook 😞

- Relativistic solution slower than the non-relativistic one
- Difference depends on the amplitude of oscillation
Limitations of Practical Approach

- $L = -mc^2 \sqrt{1 - \beta^2} - V(x)$ gives correct relativistic answers for many practical problems
- It is an ad-hoc technique
  - Not Lorentz covariant by construction
    - Time is treated separately from space
  - Lorentz transformation of Lagrangian is not given
    - Must redefine $L$ in each inertial frame
- Truly relativistic theory should respect relativity from the principle all the way up
  - Let’s see how well it works…
Lagrangian Formulation

- **Practical Approach**
  - Build a Lagrangian that reproduces 3-force in a frame
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- **Proper Approach**
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… but it quickly runs into difficulties … even for a single particle. For a system of more than one particle, it breaks down almost from the start. No satisfactory formulation for an interacting multiparticle system exists in classical relativistic mechanics except for some few special cases. 

Goldstein, p. 313
Truly Relativistic Formalism

- Hamilton’s principle
  \[ \delta I = \delta \int L \, dt = 0 \]
  - We want the action integral to be Lorentz scalar
  - Integration should not be by \( t \), but by a Lorentz-invariant variable \( \tau \) could be a good choice?
  - Lagrangian \( L \) must then be a Lorentz scalar

- Lagrange’s equation should look like
  \[ \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial u^\mu} \right) = 0 \]
  Symmetric for time and space components

- Solution is not unique. None of them perfect
  - Let’s look at one – Goldstein Section 7.10 for more
Free Lagrangian

- We try a force-free Lagrangian $\Lambda = \frac{1}{2} m u^\nu u^\nu$
  - Looks like the non-relativistic kinetic energy
  - Lorentz scalar

- Lagrange’s equation would be $\frac{d}{d\tau} \left( \frac{\partial \Lambda}{\partial u^\mu} \right) = \frac{d}{d\tau} (m u^\mu) = 0$
  - Conservation of 4-momentum
  - Time component is conservation of energy

- Energy function doesn’t give total energy, though

$$h = u^\mu \frac{\partial \Lambda}{\partial u^\mu} - \Lambda = \frac{1}{2} m u^\mu u^\mu = \frac{1}{2} m c^2$$

Conserved, but not energy
We know only one force in 4-vector form → EM

- Potential was given by $qu^\mu A_\mu$

- Lagrangian can be $\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu + qu^\mu A_\mu$

- Lagrange’s equations

$$
\frac{d}{d\tau} \left( \frac{\partial \Lambda}{\partial u^\nu} \right) - \frac{\partial \Lambda}{\partial x^\nu} = \frac{d}{d\tau} \left( m u_\nu + q A_\nu \right) - qu^\mu \frac{\partial A_\mu}{\partial x^\nu} = 0
$$

- This looks promising

4-force found last week
Limitations of Purist Approach

- We don’t know 4-force for anything but EM
  - Most real-world problems cannot be solved this way
- What to do with multi-particle system
  - Lagrangian formalism allows coordinate transformation
    - Each coordinate does not correspond to a single particle
  - Problem will be solved only when we give up the particle picture

\[ \delta I = \delta \int L d\tau \quad \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial u^\mu} \right) = 0 \]

Proper time of what?
Summary

- Constructed Lagrangian formulation
  - Practical approach provides useful tools
    - Relativistic solutions can be found for many systems
    - Not really relativistic at heart
  - Purist approach can be built only for limited cases
    - E.g. single particle in EM field
- Done with special relativity
  - Next: Hamiltonian formalism

\[ L = -mc^2 \sqrt{1 - \beta^2} - V(x) \]
\[ \Lambda = \frac{1}{2} m u_\mu u^\mu + q u^\mu A_\mu \]