Lecture 18
Hamiltonian Equations of Motion
(Chapter 8)
What’s Ahead

- We are starting Hamiltonian formalism
  - Hamiltonian equation – Today and 11/26
  - Canonical transformation – 12/3, 12/5, 12/10
  - Close link to non-relativistic QM
- May not cover Hamilton-Jacobi theory
  - Cute but not very relevant
- What shall we do in the last 2 lectures?
  - Classical chaos?
  - Perturbation theory?
  - Classical field theory?
- Send me e-mail if you have preference!
Hamiltonian Formalism

- Newtonian $\rightarrow$ Lagrangian $\rightarrow$ Hamiltonian
  - Describe same physics and produce same results
  - Difference is in the viewpoints
    - Symmetries and invariance more apparent
    - Flexibility of coordinate transformation
- Hamiltonian formalism linked to the development of
  - Hamilton-Jacobi theory
  - Classical perturbation theory
  - Quantum mechanics
  - Statistical mechanics
Lagrange’s equations for $n$ coordinates

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, \ldots, n
\]

- $n$ equations $\rightarrow$ $2n$ initial conditions
- Can we do with $1^{\text{st}}$-order differential equations?
  - Yes, but you’ll need $2n$ equations
  - We keep $q_i$ and replace $\dot{q}_i$ with something similar
  - We take the conjugate momenta $p_i \equiv \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}$

2nd-order differential equation of $n$ variables

\[ q_i(t = 0) \quad \dot{q}_i(t = 0) \]
Configuration Space

- We considered \((q_1, \ldots, q_n)\) as a point in an \(n\)-dim. space
  - Called configuration space
  - Motion of the system \(\rightarrow\)
    A curve in the config space
- When we take variations,
  we consider \(q_i\) and \(\dot{q}_i\) as independent variables
  - i.e., we have \(2n\) independent variables in \(n\)-dim. space
  - Isn’t it more natural to consider the motion in \(2n\)-dim space?

\[ q_i = q_i(t) \]
Consider coordinates and momenta as independent

State of the system is given by \((q_1, \ldots, q_n, p_1, \ldots, p_n)\)

Consider it a point in the \(2n\)-dimensional phase space

We are switching the independent variables

\((q_i, \dot{q}_i, t) \rightarrow (q_i, p_i, t)\)

A bit of mathematical trick is needed to do this
Legendre Transformation

- Start from a function of two variables \( f(x, y) \).
- Total derivative is:
  \[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy \equiv u \, dx + v \, dy
\]
- Define \( g \equiv f - ux \) and consider its total derivative:
  \[
dg = df - d(ux) = u \, dx + v \, dy - u \, dx - x \, du = v \, dy - x \, du
\]
- i.e. \( g \) is a function of \( u \) and \( y \):
  \[
  \frac{\partial g}{\partial y} = v \quad \frac{\partial g}{\partial u} = -x
  \]
  If \( f = L \) and \( (x, y) = (\dot{q}, q) \):
  \[
  L(q, q) \to g(p, q) = L - p \dot{q}
  \]
  This is what we need.
**Hamiltonian**

- **Define Hamiltonian:** \( H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t) \)

- **Total derivative is**
  \[
dH = p_i \dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt
  \]

- **Lagrange’s equations say**
  \[
  \frac{\partial L}{\partial q_i} = \ddot{q}_i
  \]

- **This must be equivalent to**
  \[
dH = \dot{q}_i dp_i - \ddot{q}_i dq_i - \frac{\partial L}{\partial t} dt
  \]

- **Putting them together gives…**
Hamilton’s Equations

- We find \( \frac{\partial H}{\partial p_i} = \dot{q}_i \) and \( \frac{\partial H}{\partial q_i} = -\dot{p}_i \) and \( \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \)
  - \( 2n \) equations replacing the \( n \) Lagrange’s equations
  - 1\textsuperscript{st}-order differential instead of 2\textsuperscript{nd}-order
  - “Symmetry” between \( q \) and \( p \) is apparent

- There is nothing new – We just rearranged equations
  - First equation links momentum to velocity
    - This relation is “given” in Newtonian formalism
  - Second equation is equivalent to Newton’s/Lagrange’s equations of motion
Quick Example

- Particle under Hooke’s law force $F = -kx$
  
  \[ L = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2 \]
  \[ p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \]

  \[ H = \dot{p} - L = \frac{m}{2} \dot{x}^2 + \frac{k}{2} x^2 \]
  \[ = \frac{p^2}{2m} + \frac{k}{2} x^2 \]

  Replace $\dot{x}$ with $\frac{p}{m}$

- Hamilton’s equations
  \[ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \]
  \[ \dot{p} = -\frac{\partial H}{\partial x} = -kx \]

Usual harmonic oscillator
Energy Function

- Definition of Hamiltonian is identical to the energy function
  \[ h(q, \dot{q}, t) = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q, \dot{q}, t) \]

- Difference is subtle: \( H \) is a function of \((q, p, t)\)

- This equals to the total energy if
  - Lagrangian is \( L = L_0(q, t) + L_1(q, t)\dot{q}_i + L_2(q, t)\dot{q}_j\dot{q}_k \)
  - Constraints are time-independent
    - This makes \( T = L_2(q, t)\dot{q}_j\dot{q}_k \)
  - Forces are conservative
    - This makes \( V = -L_0(q) \)

See Lecture 4, or Goldstein Section 2.7
If the conditions make $h$ to be total energy, we can skip calculating $L$ and go directly to $H$.

For the particle under Hooke’s law force

$$H = E = T + V = \frac{p^2}{2m} + \frac{k}{2} x^2$$

This works often, but not always.

- when the coordinate system is time-dependent
  - e.g., rotating (non-inertial) coordinate system
- when the potential is velocity-dependent
  - e.g., particle in an EM field
Particle in EM Field

For a particle in an EM field

\[ L = \frac{m}{2} \dot{x}_i^2 - q\phi + qA_i \dot{x}_i \]

We’d be done if we were calculating \( h \)

For \( H \), we must rewrite it using \( p_i = m\dot{x}_i + qA_i \)

\[ H(x_i, p_i) = \frac{(p_i - qA_i)^2}{2m} + q\phi \]

We can’t jump on \( H = E \) because of the last term, but this is in fact \( E \).
Particle in EM Field

- Hamilton’s equations are

\[
\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i - qA_i}{m}
\]

\[
\dot{p}_i = -\frac{\partial H}{\partial x_i} = q \frac{p_j - qA_j}{m} \frac{\partial A_j}{\partial x_i} - q \frac{\partial \phi}{\partial x_i}
\]

- Are they equivalent to the usual Lorentz force?

- Check this by eliminating \( p_i \)

\[
\frac{d}{dt} (m\dot{x}_i + qA_i) = q\dot{x}_i \frac{\partial A_j}{\partial x_i} - q \frac{\partial \phi}{\partial x_i}
\]

\[
\frac{d}{dt} (mv_i) = qE_i + q(v \times B)_i
\]

A bit of work
Conservation of Hamiltonian

- Consider time-derivative of Hamiltonian

\[
\frac{dH(q,p,t)}{dt} = \frac{\partial H}{\partial q} \dot{q} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial t}
\]

\[
= -\dot{p}q + \dot{q}p + \frac{\partial H}{\partial t}
\]

- Hamiltonian is conserved if it does not depend explicitly on \( t \)

- \( H \) may or may not be total energy
  - If it is, this means energy conservation
  - Even if it isn’t, \( H \) is still a constant of motion
Cyclic Coordinates

- A cyclic coordinate does not appear in $L$
  - By construction, it does not appear in $H$ either
    
    $$H(\dot{\mathbf{q}}, p, t) = \dot{q}_i p_i - L(\dot{\mathbf{q}}, \dot{q}, t)$$
  
  - Hamilton’s equation says
    
    $$\dot{p} = -\frac{\partial H}{\partial q} = 0$$
    
    Conjugate momentum of a cyclic coordinate is conserved
  
  - Exactly the same as in the Lagrangian formalism
Cyclic Example

Central force problem in 2 dimensions

\[ L = \frac{m}{2}(r^2 + r^2 \dot{\theta}^2) - V(r) \]

\[ p_r = mr \dot{r}, \quad p_\theta = mr^2 \dot{\theta} \]

\[ H = \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r) \]

θ is cyclic

\[ p_\theta = \text{const} = l \]

Hamilton’s equations

\[ \dot{r} = \frac{p_r}{m} \]

\[ \dot{p}_r = \frac{l^2}{mr^3} - \frac{\partial V(r)}{\partial r} \]

Cyclic variable drops off by itself
Going Relativistic

- Practical approach
  - Find a Hamiltonian that “works”
    - Does it represent the total energy?

- Purist approach
  - Construct covariant Hamiltonian formalism
    - For one particle in an EM field

- Don’t expect miracles
  - Fundamental difficulties remain the same
Practical Approach

- Start from the relativistic Lagrangian that “works”

\[ L = -mc^2 \sqrt{1 - \beta^2} - V(x) \]

\[ p_i = \frac{\partial L}{\partial v_i} = \frac{mv_i}{\sqrt{1 - \beta^2}} \]

\[ H = h = \sqrt{p^2c^2 + m^2c^4} + V(x) \]

- It does equal to the total energy

- Hamilton’s equations

\[ \dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i c^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{p_i}{m\gamma} \]

\[ \dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i \]
Consider a particle in an EM field

\[ L = -mc^2 \sqrt{1 - \beta^2} - q\phi(x) + q(v \cdot A) \]

- Hamiltonian is still total energy
  \[ H = m\gamma c^2 + q\phi \]
  \[ = \sqrt{m^2 \gamma^2 v^2 c^2 + m^2 c^4} + q\phi \]

- Difference is in the momentum
  \[ p_i = m\gamma v_i + qA_i \]

\[ H = \sqrt{(p - qA)^2 c^2 + m^2 c^4} + q\phi \]

Not the usual linear momentum!
Practical Approach w/ EM Field

\[ H = \sqrt{(p - qA)^2 c^2 + m^2 c^4 + q\phi} \]

- Consider \( H - q\phi \)

\[ (H - q\phi)^2 - (p - qA)^2 c^2 = m^2 c^4 \]

- It means that \( (H - q\phi, pc - qAc) \) is a 4-vector,
and so is \( (H, pc) \)

- Similar to 4-momentum \( (E/c, p) \) of a relativistic particle

- This particular Hamiltonian + canonical momentum transforms as a 4-vector

- True only for well-defined 4-potential such as EM field

\[ H = \sqrt{(p - qA)^2 c^2 + m^2 c^4 + q\phi} \]

Remember \( p \) here is not the linear momentum!
Covariant Lagrangian for a free particle

\[ \Lambda = \frac{1}{2} m \nu \mu \nu \]

\[ p^\mu = \frac{\partial \Lambda}{\partial u_\mu} = m u^\mu \]

\[ H = \frac{p_\mu p^\mu}{2m} \]

- We know that \( p^0 \) is \( E/c \)
- We also know that \( x^0 \) is \( ct \)…

Energy is the conjugate “momentum” of time

- Generally true for any covariant Lagrangian
- You know the corresponding relationship in QM
Purist Approach

- Value of Hamiltonian is
  \[ H = \frac{p_\mu p^\mu}{2m} = \frac{mc^2}{2} \]
  This is constant!

- What is important is \( H \)'s dependence on \( p^\mu \)

- Hamilton's equations
  \[
  \frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = \frac{p^\mu}{m} \quad \frac{dp^\mu}{d\tau} = -\frac{\partial H}{\partial x_\mu} = 0
  \]

- Time components are
  \[
  \frac{d(ct)}{d\tau} = \frac{E}{mc} = \gamma c \quad \frac{d(E/c)}{d\tau} = 0
  \]
  Energy definition and conservation

- 4-momentum conservation
Purist Approach w/ EM Field

- With EM field, Lagrangian becomes
  \[ \Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu + q u^\mu A_\mu \rightarrow p^\mu = m u^\mu + q A^\mu \]

- Hamilton’s equations are
  \[ \frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = \frac{p^\mu - q A^\mu}{m} \quad \frac{dp^\mu}{d\tau} = -\frac{\partial H}{\partial x_\mu} = -\left( p_\nu - q A_\nu \right) \frac{\partial A^\nu}{\partial x_\mu} \]

- A bit of work can turn them into
  \[ m \frac{du^\mu}{d\tau} = q \left( \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) u_\nu = K^\mu \]

4-force
In Hamiltonian formalism, EM field always modify the canonical momentum as
\[ p'^\mu = p^\mu + qA^\mu \]

A handy rule:

Hamiltonian with EM field is given by replacing \( p^\mu \) in the field-free Hamiltonian with \( p^\mu - qA^\mu \)

Often used in relativistic QM to introduce EM interaction
Summary

- Constructed Hamiltonian formalism
  - Equivalent to Lagrangian formalism
    - Simpler, but twice as many, equations
  - Hamiltonian is conserved (unless explicitly $t$-dependent)
    - Equals to total energy (unless it isn’t) (duh)
  - Cyclic coordinates drops out quite easily

- A few new insights from relativistic Hamiltonians
  - Conjugate of time = energy
  - $p^\mu - qA^\mu$ rule for introducing EM interaction