What We Did Last Time

- Direct Conditions
  - Necessary and sufficient for Canonical Transf.
- Infinitesimal CT
- Poisson Bracket
  - Canonical invariant
  - Fundamental PB \( [q_i, q_j] = [p_i, p_j] = 0 \), \( [q_i, p_j] = -[p_i, q_j] = \delta_{ij} \)
- ICT expressed by \( \delta u = \varepsilon [u, G] + \frac{\partial u}{\partial t} \)
- Infinitesimal time transf. generated by Hamiltonian → Hamilton’s equations

Two Points of View

- Canonical Transformation allows one system to be described by multiple sets of coordinates/momenta
  - Same physical system is expressed in different phase spaces
  - This is the static view – The system itself is unaffected

Is there a dynamic view?
Dynamic View of CT

- A system evolves with time \(q(t_0), p(t_0) \rightarrow q(t), p(t)\)
  - At any moment, \(q\) and \(p\) satisfy Hamilton’s equations
  - The time-evolution must be a Canonical Transformation!

Static View = Coordinate system is changing
Dynamic View = Physical system is moving

Infinitesimal Time CT

- Infinitesimal CT \(q(t_0), p(t_0) \rightarrow q(t + dt), p(t + dt)\)
  - We know that the generator = Hamiltonian
    \[ \frac{du}{dt} = dt[H, H] + \frac{\partial u}{\partial t} dt \]
    \[ \dot{q} = [q, H] \quad \dot{p} = [p, H] \]

Hamiltonian is the generator of the system’s motion with time

- Integrating it with time should give us the “finite” CT that turns the initial conditions \(q(t_0), p(t_0)\) into the configuration \(q(t), p(t)\) of the system at arbitrary time
- That’s a new definition of “solving” the problem

Static vs. Dynamic

- Two ways of looking at the same thing
  - System is moving in a fixed phase space
    - Hamilton’s equations \(\rightarrow\) Integrate to get \(q(t), p(t)\)
  - System is fixed and the phase space is transforming
    - ICT given by the PB \(\rightarrow\) Integrate to get CT for finite \(t\)

Equations are identical
- You’ll find yourself integrating exactly the same equations

Did we gain anything?
Conservation

- Consider an ICT generated by $G$
  - $\delta u = \varepsilon [u, G] + \frac{\partial u}{\partial t} \delta t$
- Suppose $G$ is conserved and has no explicit $t$-dependence $[G, H] = 0$
- How is $H$ (without $t$-dependence) changed by the ICT?
  $$\delta H = \varepsilon [H, G] + \frac{\partial H}{\partial t} \delta t = 0$$
  If an ICT does not affect Hamiltonian, its generator is conserved
- A transformation that does not affect $H$
  $\rightarrow$ Symmetry of the system
  $\rightarrow$ Generator of the transformation is conserved

Momentum Conservation

- Simplest example:
  - What is the ICT generated by momentum $p_i$?
    $$\delta q_j = \varepsilon [q_j, p_i] = \delta x_j \quad \delta p_j = \varepsilon [p_j, p_i] = 0$$
    - That’s a shift in $q_j$ by $\varepsilon$ $\rightarrow$ spatial translation
    - If Hamiltonian is unchanged by such shift, then $[H, p_i] = 0$
      $\rightarrow$ Momentum $p_i$ is conserved
  - This is not restricted to linear momentum

Angular Momentum

- Let’s consider a specific case: Angular momentum
  - Pick $x$-$y$-$z$ system with $z$ being the axis of rotation
    - $n$ particles’ positions given by $(x, y, z)$
  - Rotate all particles CCW around $z$ axis by $d\theta$
    - $x’ = x - yd\theta$
      $y’ = y + xd\theta$
  - Momenta are rotated as well
    - $p’_x = p_x - pd\theta$
      $p_y = p_y + p_d d\theta$
  - Generator is
    $$\varepsilon = x p_y - y p_x$$
    $$d\theta [x, G] = d\theta \varepsilon \frac{\partial G}{\partial p_y} = -y d\theta$$
    $$d\theta [p_j, G] = -d\theta \varepsilon \frac{\partial G}{\partial x_j} = -p_j d\theta$$
    etc.
Angular Momentum

- The generator \( G = x \hat{p}_y - y \hat{p}_x \) is obviously \( L_z = (\mathbf{r} \times \mathbf{p})_z \).
- i.e. the \( z \)-component of the total momentum
- Generator for rotation about an axis given by a unit vector \( \mathbf{n} \) should be \( G = L \cdot \mathbf{n} \)

- We now know generators of 3 important ICTs
  - Hamiltonian generates displacement in time
  - Linear momentum generates displacement in space
  - Angular momentum generates rotation in space

Integrating ICT

- I said we can “integrate” ICT to get finite CT
- How do we integrate \( \frac{du}{d\alpha} = \hat{u}(\alpha, G) \)?
- First, let’s rewrite it as \( du = d\alpha \hat{u}(\alpha, G) \Rightarrow \frac{du}{d\alpha} = \hat{u}(\alpha, G) \)
- We want the solution \( u(\alpha) \) as a function of \( \alpha \), with the initial condition \( u(0) = u_0 \)
- Taylor expand \( u(\alpha) \) from \( \alpha = 0 \)

\[
\begin{align*}
  u(\alpha) &= u_0 + \alpha \frac{du}{d\alpha} + \frac{\alpha^2}{2!} \frac{d^2 u}{d\alpha^2} + \frac{\alpha^3}{3!} \frac{d^3 u}{d\alpha^3} + \cdots \\
  This \ is \ [u,G]_0 \\
  What \ can \ I \ do \ with \ these?
\end{align*}
\]

- Since \( \frac{du}{d\alpha} = \hat{u}(\alpha, G) \) is true for any \( \alpha \), we can say \( \frac{d}{d\alpha} \hat{u} = \{u, G\} \)
- Now apply this operator repeatedly

\[
\begin{align*}
  \frac{d^2 u}{d\alpha^2} &= \frac{d}{d\alpha} \{u, G\} = \{\{u, G\}, G\} \Rightarrow \frac{d^2 u}{d\alpha^2} = \{\{\{u, G\}, G\}, \cdots, G\} \\
  This \ going \ back \ to \ the \ Taylor \ expansion, \\
  \begin{align*}
  u(\alpha) &= u_0 + \alpha \frac{du}{d\alpha} + \frac{\alpha^2}{2!} \frac{d^2 u}{d\alpha^2} + \frac{\alpha^3}{3!} \frac{d^3 u}{d\alpha^3} + \cdots \\
  &= u_0 + \alpha \{u, G\}_1 + \frac{\alpha^2}{2!} \{\{u, G\}, G\}_2 + \frac{\alpha^3}{3!} \{\{\{u, G\}, G\}, G\}_3 + \cdots \\
  \end{align*}
\end{align*}
\]

- Now we have a formal solution – But does it work?
Rotation CT

- Let’s integrate the ICT for rotation around $z$
  - Let me forget the particle index $i$
  - Parameter $a$ is $\theta$ in this case
  - Let’s see how $x$ changes with $\theta$
    \[ x(\theta) = x_0 + \theta \mathcal{G}(0) + \frac{\theta^2}{2!} [x, \mathcal{G}] + \frac{\theta^3}{3!} [[x, \mathcal{G}], \mathcal{G}] + \cdots \]
  - Evaluate the Poisson Brackets
    \[
    [x, \mathcal{G}] = -y \\
    [[x, \mathcal{G}], \mathcal{G}] = -x \\
    [[[x, \mathcal{G}], \mathcal{G}], \mathcal{G}] = x
    \]
    Repeats after this
  - Where does this lead us?

Rotation CT

\[
\begin{align*}
x(\theta) &= x_0 + \theta \mathcal{G}(0) + \frac{\theta^2}{2!} [x, \mathcal{G}] + \frac{\theta^3}{3!} [[x, \mathcal{G}], \mathcal{G}] + \cdots \\
&= x_0 - \theta y_0 - \frac{\theta^2}{2!} y_0 + \frac{\theta^3}{3!} y_0 + \frac{\theta^4}{4!} y_0 - \cdots \\
&= x_0 \left(1 - \frac{\theta^2}{2!} \theta^4 \frac{\theta^6}{4!} \cdots \right) - y_0 \left(\theta - \frac{\theta^3}{3!} \theta^5 \frac{\theta^7}{5!} \cdots \right) \\
&= x_0 \cos \theta - y_0 \sin \theta
\end{align*}
\]
- Similarly

Free Fall

- An object is falling under gravity
  - Hamiltonian is $H = \frac{p^2}{2m} + mgz$
  - Integrate the time ICT
    \[
    z(t) = z_0 + \int z \text{d}t + \frac{t^2}{2!} [z, H] + \frac{t^3}{3!} [[[z, H], H], H] + \cdots \\
    [z, H] = \frac{p}{m} \\
    [[[z, H], H], H] = -g \\
    z(t) = z_0 + \frac{p_0}{m} - \frac{g t^2}{2}
    \]
    It's easier than it looked
Infinitesimal Rotation

- ICT for rotation is generated by $G = L \cdot n$
- We’ve studied infinitesimal rotation in Lecture 8
- Infinitesimal rotation of $d\theta$ about $n$ moves a vector $r$ as $dr = n d\theta \times r$
- Compare the two expressions $dr = d\theta [r, L \cdot n] = n d\theta \times r \quad [r, L \cdot n] = n \times r$
- Equation $[r, L \cdot n] = n \times r$ holds for any $r$ that rotates together with the system
- Several useful rules can be derived from this

Scalar Products

- Consider a scalar product $a \cdot b$ of two vectors
- Try to rotate it
  \[
  a \cdot b, L \cdot n = a \cdot [b, L \cdot n] + b \cdot [a, L \cdot n]
  = a \cdot (n \times b) + b \cdot (n \times a)
  = a \cdot (n \times b) + a \cdot (b \times n)
  = 0
  \]
- Obvious: scalar product doesn’t change by rotation
- Also obvious: length of any vector is conserved

Angular Momentum

- Try with $L$ itself $[L, L \cdot n] = n \times L$
- $x$-$y$-$z$ components are
  \[
  \begin{align*}
  [L_x, L_x] &= 0, & [L_x, L_y] &= L_z, & [L_x, L_z] &= -L_y \\
  [L_y, L_x] &= -L_z, & [L_y, L_y] &= 0, & [L_y, L_z] &= L_x \\
  [L_z, L_x] &= L_y, & [L_z, L_y] &= -L_x, & [L_z, L_z] &= 0
  \end{align*}
  \]
- These relationships are well-known in QM
- They tell us two rather interesting things…
Angular Momentum

- Imagine two conserved quantities $A$ and $B$
  \[ [A, H] = [B, H] = 0 \]
- How does $[A, B]$ change with time?
  \[ [[A, B], H] - [B, [H, A]] - [H, [A, B]] = 0 \]
  - Poisson bracket of two conserved quantities is conserved
- Now consider $[L_i, L_j] = \epsilon_{ijk} L_k$
  - If 2 components of $L$ are conserved, the 3rd component must
  - Total vector $L$ is conserved

Angular Momentum

- Remember the Fundamental Poisson Brackets?
  \[ \{q_i, q_j\} = \{p_i, p_j\} = 0 \]
  \[ \{q_i, p_j\} = -\delta_{ij} \]
  - PB of two canonical momenta is 0
- Now we know $[L_i, L_j] = \epsilon_{ijk} L_k$
  - Poisson brackets between $L_x, L_y, L_z$ are non-zero
  - Only 1 of the 3 components of the angular momentum can be a canonical momentum
  - On the other hand, $[F_j, L_j] = 0$; so $|L|$ may be a canonical momentum
- QM: You may measure $|L|$ and, e.g., $L_z$ simultaneously, but not $L_x$ and $L_y$, etc.

Phase Volume

- Static view: CT moves a point in one phase space to a point in another phase space
- Dynamic view: CT moves a point in one phase space to another point in the same space
- If you consider a set of points, CT moves a volume to another volume, e.g.
Phase Volume

- Easy to calculate the Jacobian for 1-dimension
  \[ dQdP = |\mathbf{M}| dqdp \]
  where \( \mathbf{M} = \begin{bmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{bmatrix} \)
  \[ |\mathbf{M}| = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} = [Q,P] = 1 \]
  \[ dQdP = dqdp \]
  - i.e., volume in 1-dim. phase space is invariant
  - This is true for \( n \)-dimensions
    - Goldstein proves it using symplectic approach

Volume in Phase Space is a Canonical Invariant

Harmonic Oscillator

- We’ve seen it in the oscillator example (Lecture 21)

\[ q^2 + \frac{p^2}{m} + \frac{2\pi E}{m\omega} = \frac{\pi E}{\omega} \]
- One cycle draws the same area in both spaces
- That’s static view

Dynamic View

- Consider many particles moving independently
  - e.g., ideal gas molecules in a box
  - They obey the same EoM independently
  - Can be represented by multiple points in one phase space
  - They move with time \( \Rightarrow CT \)
Imagine ideal gas in a cylinder with movable piston. Each molecule has its own position and momentum; they fill up a certain volume in the phase space. What happens when we compress it? Compress slowly. Extra momenta $\rightarrow$ Gas gets hotter!

Liouville’s Theorem

- The phase volume occupied by a group of particles (ensemble in stat. mech.) is conserved
  - Thus the density in phase space remains constant with time
  - Known as Liouville’s theorem
  - Theoretical basis of the 2nd law of thermodynamics
- This holds true when there are large enough number of particles so that the distribution may be considered continuous
  - More about this in Physics 181

Summary

- Introduced dynamic view of Canonical Transf.
  - Hamiltonian is the generator of the motion with time
  - Symmetry of the system $\rightarrow$ Hamiltonian unaffected by the generator $\rightarrow$ Generator is conserved
- How to integrate infinitesimal transformations
  - $u(\alpha) = u + \alpha u' + \frac{\alpha^2}{2!}[u,G]_h + \frac{\alpha^3}{3!}[[[u,G],G],G]_h + \ldots$
- Discussed infinitesimal rotation
  - Angular momentum $\rightarrow$ QM
  - Invariance of the phase volume