Lecture 23
Continuous Systems and Fields
(Chapter 13)
Where Are We Now?

- We’ve finished all the essentials
  - Final will cover Lectures 1 through 22
- Last two lectures: Classical Field Theory
  - Start with wave equations, similar to Physics 15c
  - Do it with Lagrangian, and maybe with Hamiltonian
  - Go into relativistic field theory
- Not enough time to discuss everything
  - Let’s see how much we can do
    - And take it easy!
An infinite elastic rod is vibrating longitudinally

Model this with a chain of masses and springs

\[ \eta_i \text{ relative to equilibrium} \]

\[ m \quad k \]

\[ i \text{-th mass's position is } \eta_i \text{ relative to equilibrium} \]

\[ T = \sum_i \frac{1}{2} m \dot{\eta}_i^2 \]

\[ V = \sum_i \frac{1}{2} k (\eta_{i+1} - \eta_i)^2 \]

Let's build the Lagrangian
Lagrangian

- Lagrangian is

\[ L = \sum_i \frac{1}{2} \left[ m \ddot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2 \right] \]

Rearrange a little

\[ = \sum_i \frac{1}{2} \left[ \frac{m}{\Delta x} \ddot{\eta}_i^2 - k \Delta x \left( \frac{\eta_{i+1} - \eta_i}{\Delta x} \right)^2 \right] \Delta x \]

- \( m/\Delta x \) is the linear density \( \mu \) (mass/unit length)
- \( k\Delta x \) is the elastic modulus \( K \) (force/fractional elongation)

- Think about Hooke’s law

\[ F = -k\Delta L = -K \frac{\Delta L}{L} \]

It’s not Young’s modulus

- How much the spring is stretched relative to its natural length

- \( \mu \) and \( K \) remain constant as we shrink \( \Delta x \to 0 \)
Continuous Limit

Now we have

\[ L = \sum_i \frac{1}{2} \left[ \mu \dot{\eta}_i^2 - K \left( \frac{\eta_{i+1} - \eta_i}{\Delta x} \right)^2 \right] \Delta x \]

Re-label \( \eta_i \) with the equilibrium position \( x \)

\[ \eta_i \rightarrow \eta(x) \]

Shrink!

\[ \Delta x \rightarrow 0 \quad \int \frac{1}{2} \left[ \mu \dot{\eta}^2 - K \left( \frac{d\eta}{dx} \right)^2 \right] dx \]

Lagrangian per unit length
Lagrangian Density

- We can write the Lagrangian as

\[ L = \int \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right] dx \equiv \int \mathcal{L} dx \]

- \( \mathcal{L} \) is the Lagrangian density in 1-dimension

- We may generally extend this to 3-dimensions

\[ L = \iiint \mathcal{L} \, dx \, dy \, dz \quad \text{where} \quad \mathcal{L} = \frac{1}{2} \left[ \rho \left( \frac{d\eta}{dt} \right)^2 - Y \left( \frac{d\eta}{dx} \right)^2 \right] \]

- \( \rho \) is the volume density \( \mu/A \) (\( A \) is the rod’s cross section)
- \( Y \) is Young’s modulus \( K/A \)
Lagrange’s Equations

First, start from

\[ L = \sum_i \frac{1}{2} \left[ \mu \ddot{\eta}_i^2 - K \left( \frac{\eta_{i+1} - \eta_i}{\Delta x} \right)^2 \right] \Delta x \]

Do the usual Lagrange’s equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}_i} \right) - \frac{\partial L}{\partial \eta_i} = \left[ \mu \ddot{\eta}_i - K \frac{\eta_{i+1} - \eta_i}{\Delta x} + K \frac{\eta_i - \eta_{i-1}}{\Delta x} \right] \Delta x = 0 \]

Shrink \( \Delta x \)

\[ \mu \ddot{\eta}_i - K \frac{d^2 \eta}{dx^2} = 0 \]

That’s wave equation with velocity \( v = \sqrt{\frac{K}{\mu}} \)

We want to get this from the continuous Lagrangian
Lagrange’s Equations

- In the discrete case, we had
  \[ \frac{\partial L}{\partial \eta_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}_i} \right) = 0 \]
  - \( \eta_i \) became \( \eta(x) \)
  - Simple analogy gives
    \[ \frac{\partial L}{\partial \eta(x)} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}(x)} \right) = 0 \]

- But this doesn’t work
  - We must go back to Hamilton’s Principle
    \[ \delta I = \delta \int_1^2 L dt = \delta \int_1^2 \int L dx dt = 0 \]
Hamilton’s Principle

- Our Lagrangian density is
  \[ \mathcal{L} = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right] \]
  - Let’s get general
  - \( \mathcal{L} \) may depend on \( \eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t \)

- We need the “path” of \( \eta \) and its variation
  \[ \eta(x, t; \alpha) = \eta(x, t; 0) + \alpha \zeta(x, t) \]
  Will make \( \alpha \rightarrow 0 \)

- Set variation to zero at the boundaries
  \[ \zeta(x, t_1) = \zeta(x, t_2) = \zeta(x_1, t) = \zeta(x_2, t) = 0 \]

- OK, let’s work…

- Don’t really matter for the infinite rod
Hamilton’s Principle

\[
\frac{dI}{d\alpha} = \frac{d}{d\alpha} \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L}\left(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t\right) dx dt
\]

\[
= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left\{ \frac{\partial \mathcal{L}}{\partial \eta} \frac{d\eta}{d\alpha} + \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dx}} \frac{d}{dx} \frac{d\eta}{d\alpha} + \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dt}} \frac{d}{dt} \frac{d\eta}{d\alpha} \right\} dx dt
\]

\[
= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left\{ \frac{\partial \mathcal{L}}{\partial \eta} \frac{d\eta}{d\alpha} - \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dx}} \right) - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dt}} \right) \right\} \frac{d\eta}{d\alpha} dx dt
\]

Hamilton’s Principle gives

\[
\left( \frac{dI}{d\alpha} \right)_{\alpha=0} = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left\{ \frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dx}} \right) - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dt}} \right) \right\} \zeta(x, t) dx dt = 0
\]
Lagrange’s Equation

- Lagrange’s equation for the 1-dim problem is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \frac{d\eta}{dt}} \right) + \frac{d}{dx} \left( \frac{\partial L}{\partial \frac{d\eta}{dx}} \right) - \frac{\partial L}{\partial \eta} = 0
\]

- Let’s try it with

\[
L = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right]
\]

\[
\frac{d}{dt} \left( \mu \frac{d\eta}{dt} \right) - \frac{d}{dx} \left( K \frac{d\eta}{dx} \right) = \mu \frac{d^2\eta}{dt^2} - K \frac{d^2\eta}{dx^2} = 0
\]

Yes, the right wave equation
3-D Version

Easy to guess how it should look like in 3-dim.

\[ \mathcal{L} = \mathcal{L}(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dy}, \frac{d\eta}{dz}, x, y, z, t) \]

\[ I = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \mathcal{L}(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dy}, \frac{d\eta}{dz}, x, y, z, t) dx dy dz dt \]

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dt}} \right) + \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dx}} \right) + \frac{d}{dy} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dy}} \right) + \frac{d}{dz} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dz}} \right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0 \]

Symmetric between time and space

→ Hope for relativistic formalism

Will look into this in the next lecture
Multi-Component Field

- I defined $\eta$ as the displacement along $x$ axis
- General 3-dim. vibration may be in any direction
  \[ \eta = (\eta_x, \eta_y, \eta_z) \]
- We are now dealing with 3 functions of space and time

\[
\mathcal{L} = \mathcal{L} \begin{pmatrix}
\eta_x, \frac{d\eta_x}{dx}, \frac{d\eta_x}{dy}, \frac{d\eta_x}{dz}, \frac{d\eta_x}{dt}, \\
\eta_y, \frac{d\eta_y}{dx}, \frac{d\eta_y}{dy}, \frac{d\eta_y}{dz}, \frac{d\eta_y}{dt}, \\
\eta_z, \frac{d\eta_z}{dx}, \frac{d\eta_z}{dy}, \frac{d\eta_z}{dz}, \frac{d\eta_z}{dt}, \\
(x, y, z, t)
\end{pmatrix}
\]

This is getting really tedious
Shorthand Notation

- Let’s use indices \((0,1,2,3)\) instead of \(t, x, y, z\)
  - Similar to what we did in relativity
- We need quantities like \(\eta_i\)
  \[
  \frac{d\eta_i}{dx_\mu}, \quad \frac{d^2\eta_i}{dx_\mu dx_\nu}
  \]
- Let’s get lazy
  \[
  \eta_{\rho,\mu} \equiv \frac{d\eta_\rho}{dx_\mu}, \quad \eta_{\rho,\mu\nu} \equiv \frac{d^2\eta_\rho}{dx_\mu dx_\nu}
  \]
  and \(\eta_{\mu} \equiv \frac{d\eta}{dx_\mu}\) etc.
- We can write, e.g.
  \[
  \mathcal{L} = \mathcal{L}(\eta_\rho, \eta_{\rho,\mu}, x_\mu)
  \]
  \[
  \frac{d}{dx_\mu} \left( \frac{\partial \mathcal{L}}{\partial \eta_{\rho,\mu}} \right) - \frac{\partial \mathcal{L}}{\partial \eta_\rho} = 0
  \]
Conservation Laws

- Let’s try what we did with the energy function
  - Consider the total derivative of the Lagrangian density

\[ \mathcal{L}(\eta, \eta_{,\mu}, x_{\mu}) \rightarrow \frac{d \mathcal{L}}{dx_{\mu}} = \frac{\partial \mathcal{L}}{\partial \eta} \eta_{,\mu} + \frac{\partial \mathcal{L}}{\partial \eta_{,\nu}} \eta_{,\mu \nu} + \frac{\partial \mathcal{L}}{\partial x_{\mu}} \]

- Using Lagrange’s equations:

\[ \frac{d}{dx_{\mu}} \left( \frac{\partial \mathcal{L}}{\partial \eta_{,\mu}} \right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0 \]

\[ \frac{d \mathcal{L}}{dx_{\mu}} = \frac{d}{dx_{\nu}} \left( \frac{\partial \mathcal{L}}{\partial \eta_{,\nu}} \right) \eta_{,\mu} + \frac{\partial \mathcal{L}}{\partial \eta_{,\nu}} \eta_{,\mu \nu} + \frac{\partial \mathcal{L}}{\partial x_{\mu}} \]

This is \( \frac{d \eta_{,\mu}}{dx_{\nu}} \)
Stress-Energy Tensor

- We got

\[
\frac{d}{dx^v} \left( \frac{\partial \mathcal{L}}{\partial \eta_{,\mu}} \eta_{,\mu} - \mathcal{L} \delta^{\mu\nu} \right) = -\frac{\partial \mathcal{L}}{\partial x^\mu} \equiv T_{\mu\nu}
\]

- NB: \( T_{\mu\nu} \) is not a tensor in the relativistic sense

- Suppose \( \mathcal{L} \) does not depend explicitly on \( x^\mu \)
  - For \( \mu = 1, 2, 3 \), that means no external force
  - For \( \mu = 0 \), that means no source/sink of energy

\[
\frac{dT_{\mu\nu}}{dx^v} = 0
\]

What does this “conservation” condition mean?
Divergence of S-E Tensor

- The condition \( \frac{dT_{\mu \nu}}{dx_{\nu}} = 0 \) has a form of divergence

\[
\frac{dT_{\mu \nu}}{dx_{\nu}} = \frac{dT_{\mu 0}}{dt} + \frac{dT_{\mu i}}{dx_i} = \frac{dT_{\mu 0}}{dt} + \nabla \cdot T_{\mu} = 0
\]

- Integrate over a fixed volume \( V \) and use Gauss’s Law

\[
\frac{d}{dt} \int T_{\mu 0} dV = -\int \nabla \cdot T_{\mu} dV = -\int T_{\mu} \cdot dS
\]

This vector represents the "flow"

- What escapes from the surface

- Now we need to know what \( T_{\mu 0} \) and \( T_{\mu} \) are
Energy Density

- First consider \( T_{00} = \frac{\partial \mathcal{L}}{\partial \eta} \dot{\eta} - \mathcal{L} \)

- Looks just like the energy function, doesn’t it?

- Think about the 1-dim. elastic rod example

\[
\mathcal{L} = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right]
\]

\( T_{00} = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 + K \left( \frac{d\eta}{dx} \right)^2 \right] \)

- \( T_0 \) should be the energy flow

\[
T_{01} = \frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dx}} \dot{\eta} = -K \frac{d\eta}{dx} \dot{\eta}
\]

is it?

\[
T_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial \eta_{,\mu}} \eta_{,\nu} - \mathcal{L} \delta_{\mu\nu}
\]
Energy Current Density

- Consider a small piece
  - It’s stretched by
    \[ \eta(x + dx) - \eta(x) = \frac{d\eta}{dx} dx \]
  - This gives the Hooke’s law force
    \[ F = -K \frac{d\eta}{dx} \]
  - The work done by this piece to the next piece is
    \[ F\dot{\eta} = -K \frac{d\eta}{dx} \dot{\eta} \]
    equals to
    \[ T_{01} = -K \frac{d\eta}{dx} \dot{\eta} \]
Momentum Density

First consider \( T_{i0} = \frac{\partial L}{\partial \dot{\eta}_i} \frac{d\eta}{dx} \)

Again with the 1-dim. elastic rod example

\[
\mathcal{L} = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right]
\]

This isn’t so obvious…
Momentum Density

- How much mass is there between $x$ and $x + dx$?
  - $\mu dx$ to the zeroth order
  - To the first order
    $$\mu \left(1 - \frac{d\eta}{dx}\right) dx$$
  - It’s velocity is $\dot{\eta}$, so the momentum is
    $$\mu \left(1 - \frac{d\eta}{dx}\right) \frac{d\eta}{dt} dx$$
  - Density of excess momentum is
    $$-\mu \frac{d\eta}{dt} \frac{d\eta}{dx} = -T_{10}$$
- $-T_{10}$ may be considered as the momentum density

$$T_{10} = \mu \frac{d\eta}{dt} \frac{d\eta}{dx}$$
We can interpret the stress-energy tensor $T_{\mu\nu}$ as

- $T_{00}$ = energy density
- $T_{0i}$ = energy current density
- $T_{i0}$ = momentum density
- $T_{ij}$ = momentum current density

The divergence condition $\frac{dT_{\mu\nu}}{dx_\nu} = 0$ represents conservation of energy and momentum.
Summary

- Built Lagrangian formalism for continuous system
  - Lagrangian $L = \iiint L \, dx dy dz$
  - Lagrange’s equation $\frac{d}{dx_\mu} \left( \frac{\partial L}{\partial \eta_{,\mu}} \right) - \frac{\partial L}{\partial \eta} = 0$
  - Derived simple wave equation

- Energy and momentum conservation given by the energy-stress tensor
  - Conservation laws take the form of $(\text{time derivative}) = (\text{flux into volume})$
    - $T_{\mu\nu} = \frac{\partial L}{\partial \eta_{,\nu}} \eta_{,\mu} - L \delta_{\mu\nu}$
    - $\frac{dT_{\mu\nu}}{dx_\nu} = 0$