Mechanics
Physics 151

Lecture 24
Continuous Systems and Fields
(Chapter 13)
What We Did Last Time

- Built Lagrangian formalism for continuous system
  - Lagrangian $L = \iiint L \, dx \, dy \, dz$
  - Lagrange’s equation
  - Derived simple wave equation

- Energy and momentum conservation given by the energy-stress tensor
  - Conservation laws take the form of (time derivative) = (flux into volume)
  - Ran out of time here → See Goldstein 13.3 if interested
  - Today’s lecture doesn’t use it
Hamiltonian Formalism

- For a discrete system, we define conjugate momenta
  \[ p_i = \frac{\partial L}{\partial \dot{q}_i} \]
  Then
  \[ H = p_i \dot{q}_i - L \]

- For a continuous system,
  \[ L = \iiint L(\eta, \dot{\eta}, \frac{d\eta}{dx_i}, t, x_i) \, dx \, dy \, dz \]

- Momentum should be
  \[ \pi(t, x_i) = \frac{\partial L}{\partial \dot{\eta}} \]

- Hamiltonian
  \[ H = \iiint H \, dx \, dy \, dz \] where \[ H = \pi \dot{\eta} - L \]

- Let’s see how this works…
Consider the 1-dim elastic rod again

Lagrangian density

\[ L = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right] \]

\[ \pi = \frac{\partial L}{\partial \dot{\eta}} = \mu \dot{\eta} \]

\[ H = \pi \dot{\eta} - L = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 + K \left( \frac{d\eta}{dx} \right)^2 \right] \]

Wait! What am I going to do with this term?
Hamiltonian Formalism

- Hamiltonian formalism treats time as special
  - Because of the way momentum is defined \( \pi = \partial L / \partial \dot{\eta} \)

- Natural structure of classical field theory is symmetric between time and space
  - At least in Lagrangian formalism
  - Hamiltonian is not so useful as in the case of discrete systems

- Quantum field theory is built primarily on Lagrangian
  - c.f. Non-relativistic QM is almost all Hamiltonian
Consider an elastic rod with finite length $L$.

At a given moment $t$, we can Fourier transform $\eta(x, t)$:

$$\eta(x, t) = \sum_{n=0}^{\infty} q_n(t) \sin \frac{\pi nx}{L} = \sum_{n=0}^{\infty} q_n(t) \sin k_n x$$

Or, using the complex form,

$$\eta(x, t) = \sum_{n=0}^{\infty} q_n(t) e^{i \frac{\pi nx}{L}} = \sum_{n=0}^{\infty} q_n(t) e^{i k_n x}$$

Assuming $\eta(0) = \eta(L) = 0$

$q_n(t)$ is a complex function

Re() assumed
Fourier Transformation

- What happens to the Lagrangian?

\[ \mathcal{L} = \frac{1}{2} \left[ \mu \left( \frac{d\eta}{dt} \right)^2 - K \left( \frac{d\eta}{dx} \right)^2 \right] \]

\[ = \frac{1}{2} \left[ \mu \sum_n \dot{q}_n \sin k_n x \sum_m \dot{q}_m \sin k_m x - K \sum_n q_n k_n \cos k_n x \sum_m q_m k_m \cos k_m x \right] \]

- Integrate with \( x \) and use \( \int_0^L \sin k_n x \sin k_m x dx = \frac{L}{2} \delta_{nm} \) etc.

\[ \int_0^L \mathcal{L} dx = \frac{L}{2} \left[ \mu \sum_n \frac{\dot{q}_n^2}{2} - K \sum_n k_n^2 \frac{q_n^2}{2} \right] = \frac{L}{2} \sum_n \left[ \frac{\mu}{2} \frac{\dot{q}_n^2}{2} - \frac{K k_n^2}{2} q_n^2 \right] \]

What does this look like?

\[ \eta(x, t) = \sum_{n=0}^{\infty} q_n(t) \sin k_n x \]
Harmonic Oscillators

- The Lagrangian represents an infinite array of independent harmonic oscillators
  - Angular frequencies are \( \omega_n = \sqrt{\frac{Kk^2_n}{\mu}} = v_kn \)
  - Wave velocity
  - Wavenumber

- Vibration of continuous system can be decomposed into a set of discrete oscillators
  - True for any linear system
    - Lagrangian density must be 2nd order homogeneous function of the field’s derivatives
    - Small oscillation around equilibrium always OK

\[
\frac{L}{2} \sum_n \left[ \frac{\mu}{2} \dot{q}_n^2 - \frac{Kk^2_n}{2} q_n^2 \right]
\]
Phonons

- Harmonic oscillator in QM has discrete energy levels
  - Possible values of $E$ are $E = (m + \frac{1}{2})\hbar \omega$ (where $m = 0, 1, 2, \ldots \infty$)
  - What does this mean for the continuous system?

- $\eta(x,t)$ is a superposition of sine waves with different $k$
  - Each mode is a harmonic oscillator
    
    \[ \omega_n = v k_n \quad \Rightarrow \quad E_n = (m_n + \frac{1}{2})\hbar \omega_n \]

  - Vibration energy comes in small-but-finite pieces of $\hbar \omega_n$
  - As if it’s a bunch of particles

- Vibration can be seen as particles
  - Called phonons in the case of mechanical vibration
Other Examples?

- Linear fields $\rightarrow$ Harmonic oscillators $\rightarrow$ Particles
  - We know an excellent example: Electromagnetic field
  - Corresponding particle = photon
    - Photoelectric effect tells us $E = \hbar \omega$
- Is it possible that all particles are quantized field?
  - For a particle of mass $m$, $E = \sqrt{m^2 c^4 + p^2 c^2}$
  - Make correspondence with a harmonic oscillator
    - $\hbar^2 \omega^2 = m^2 c^4 + p^2 c^2$ $\rightarrow$ $\omega^2 = \frac{m^2 c^4}{\hbar^2} + k^2 c^2$
  - But first of all, the field must satisfy relativity

Must satisfy this dispersion relation
Relativistic Field Theory

- We had difficulty with relativity and multi-particles
  - Each particle’s EoM looked like \( \frac{dp_s}{d\tau_s} = K_s \)
  - When combined, we didn’t know whose time to use
- With field like \( \eta(x,t) \), time is just another parameter
  - Action integral and Lagrange’s equations look symmetric for time and space
- Can we just call \( x^0 = ct \) and call it done?
  - Almost…

\[
\int \int \int \int L d\mathbf{x} dy dz dt
\]
Lagrangian Density

- Everything depends on the action integral
  - It must be Lorentz invariant \( \rightarrow \) All the equations will follow
  - Write it as \[ I = \int \mathcal{L} \, dx^0 dx^1 dx^2 dx^3 \]
  - The volume element \( dx^0 dx^1 dx^2 dx^3 \) is Lorentz invariant
    - Because \( \text{det}(L^\mu_\nu) = 1 \) for any Lorentz tensor

Lagrangian density \( \mathcal{L} \) must be a Lorentz scalar

- You must construct \( \mathcal{L} \) using covariant quantities
  - Your field may be scalar (\( \eta \)) or 4-vector (\( \eta_\mu \)) or tensor…
  - You combine them so that the product is a scalar
We derived Lagrange’s equation from Hamilton’s principle for continuous field in the last lecture.

Derivation is unchanged $\Rightarrow$ Same equations hold

$$\delta I = \delta \int \mathcal{L} \, dx^0 \, dx^1 \, dx^2 \, dx^3 = 0$$

$$\frac{d}{dx^\mu} \left( \frac{\partial \mathcal{L}}{\partial \eta_{\rho,\mu}} \right) - \frac{\partial \mathcal{L}}{\partial \eta_{\rho}} = 0$$

Note

$$\frac{d}{dx^0} \left( \frac{\partial \mathcal{L}}{\partial \eta_{\rho,0}} \right) = \frac{d}{d(ct)} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta_{\rho}}{d(ct)}} \right) = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \frac{d\eta_{\rho}}{dt}} \right)$$

Didn’t change this term

Ready to look at an easy example…
Scalar Field

- The simplest field is a scalar field $\phi$
  - Lagrangian density may be a function of $L(\phi, \phi, x)$
  - For free field, $L$ has no explicit dependence on $x$.
  - Only a few scalar quantities can be formed
  - Try $L = \phi, x, x - \mu_0^2 \phi^2$

\[
\frac{d}{dx^\nu} \left( \frac{\partial L}{\partial \phi_{,\nu}} \right) - \frac{\partial L}{\partial \phi} = \frac{d}{dx^\nu} \left( 2\phi_{,\nu} \right) + 2\mu_0^2 \phi = 0
\]

- What kind of field is this?

Known as Klein-Gordon equation

\[
\frac{d^2 \phi}{dx^\nu dx^\nu} + \mu_0^2 \phi = 0
\]
Klein-Gordon Equation

- Let’s do Fourier in space volume $V$
  \[ \phi = \sum_k q_k e^{i k \cdot r} \text{ where } q_k = \frac{1}{V} \int \phi e^{-i k \cdot r} dV \]
  \[ k \text{ takes all the values that satisfy the boundary condition} \]

- Klein-Gordon equation is then
  \[ \frac{d^2 \phi}{dx^\nu dx_\nu} + \mu_0^2 \phi = \frac{d^2 \phi}{c^2 dt^2} - \nabla^2 \phi + \mu_0^2 \phi = \sum_k \left\{ \frac{1}{c^2} \ddot{q}_k + k^2 q_k + \mu_0^2 q_k \right\} e^{i k \cdot r} = 0 \]

- For each mode $k$, \[ \frac{1}{c^2} \ddot{q}_k + k^2 q_k + \mu_0^2 q_k = 0 \] Harmonic oscillator!

- Dispersion relation is \[ \omega_k^2 = c^2 (k^2 + \mu_0^2) \]

- Corresponds to a particle with a finite mass \[ m = \frac{\hbar}{c} \mu_0 \]
The Field – What is It?

- $\mathcal{L} = \phi\ddot{\phi} - \mu_0 \phi^2$ gives particles with mass $m = \frac{\hbar}{c} \mu_0$

- OK, but what is the field $\phi$ itself?
  - Vibration of elastic material $\rightarrow$ Phonons
  - Vibration of electromagnetic field $\rightarrow$ Photons

- The field $\phi$ doesn’t have to be “physical”
  - It “exists” only in the sense that quantized excitation of $\phi$ are physical (particles)
  - QM calls it wave function, whose $(amplitude)^2$ is interpreted as the probability of a particle being there
    - Still an indirect definition of “existence”
Vector Field

- Field can be more complicated than scalar
  - How about a 4-vector, for example?
  - Such field represents particles with spins
    - 4-vector field \( \rightarrow \) Particles with spin = 1

- Electromagnetic field is an obvious example
  - Corresponding particle is photon, with spin 1
  - Recall \( A_\mu = (\phi/c, A) \) is a 4-vector
  - Connection with \( \mathbf{E} \) and \( \mathbf{B} \)

\[
F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} =
\begin{bmatrix}
0 & -E_x/c & -E_y/c & -E_z/c \\
E_x/c & 0 & -B_z & B_y \\
E_y/c & B_z & 0 & -B_x \\
E_z/c & -B_y & B_x & 0
\end{bmatrix}
\]
Electromagnetic Field

- EM field interacts with charge
  - Maxwell's equations
  - In terms of \( F^{\mu\nu} \),
  - Defining 4-current as \( j^\mu = (\rho c, j) \),

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial F^{0\nu}}{\partial x^\nu} &= -\frac{\nabla \cdot \mathbf{E}}{c} = -\frac{\rho}{c \varepsilon_0} \\
\frac{\partial F^{i\nu}}{\partial x^\nu} &= -\left( \nabla \times \mathbf{B} \right)^i + \frac{1}{c^2} \frac{\partial E^i}{\partial t} = -\mu_0 j^i
\end{align*}
\]

- Let's pick a unit in which \( \mu_0 = 1 \)
- What's the Lagrangian?
To build $\mathcal{L}$, we can use $A^\mu$, $F^{\mu\nu}$ and $j^\mu$

Easy to find $\mathcal{L}$ that works:

$$\mathcal{L} = \frac{F^{\lambda\rho} F_{\lambda\rho}}{4} + j^\lambda A_\lambda$$

Field equation is

$$\frac{d}{dx^\nu} \left( \frac{\partial \mathcal{L}}{\partial (A_{\mu,v})} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{d}{dx^\nu} F^{\lambda\rho} \frac{\partial F_{\lambda\rho}}{\partial A_{\mu,v}} - j^\mu$$

$$= \frac{d}{dx^\nu} -F^{\mu\nu} + F^{\nu\mu} \frac{2}{2} - j^\mu$$

$$= -\frac{dF^{\mu\nu}}{dx^\nu} - j^\mu = 0$$

What we wanted
Free EM Field

- Does it satisfy the usual wave equation?
  - For free field \((j^\mu = 0)\), the field equation reduces to
    \[
    \frac{d}{dx^\nu} \left( \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) = \frac{\partial^2 A^\nu}{\partial x_\mu \partial x^\nu} - \frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = 0
    \]
    - This doesn’t give you the usual plane waves etc.
  - Problem: Given \(E\) and \(B\), \(A^\mu\) is not uniquely defined
    - Extra condition to fix this ambiguity
  - Impose Lorentz gauge condition
    \[
    \frac{\partial A^\mu}{\partial x^\mu} = 0 \quad \text{EM waves with } v = c
    \]
Gauge Conditions

- We may add a gradient of any function \( \Lambda \) to \( A^\mu \)

\[
A'_{\mu} = A^\mu + \frac{\partial \Lambda}{\partial x_\mu} \quad \quad \quad F'_{\mu\nu} = \frac{\partial A'_{\nu}}{\partial x_\mu} - \frac{\partial A'_{\mu}}{\partial x_\nu} = F^{\mu\nu} + \frac{\partial^2 \Lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 \Lambda}{\partial x_\nu \partial x_\mu} = F^{\mu\nu}
\]

- \( A^\mu \) is not fully specified without a gauge condition
- You’ve probably seen Coulomb gauge in Physics 15b

\[ \nabla \cdot A = 0 \]

- This is not Lorentz invariant
- Natural relativistic extension is the Lorentz gauge \( \partial_\mu A^\mu = 0 \)
- All gauge conditions give you same physics
  - Some are easier than the others to solve
Relativistic Field Theory

- Classical field theory can be made relativistic
  - Not very difficult – although I omitted many subtleties…
- Lagrangian density $\mathcal{L}$ must be a Lorentz scalar
  - Built using covariant fields and currents
  - This limits the possible forms of $\mathcal{L}$
    - Guided physicists toward correct picture of Nature
- Quantization of the field produces particles
  - Fourier transformation $\rightarrow$ Harmonic oscillators
  - Quantum field theory has enjoyed great success in describing elementary particles and their interactions
We’ve come a long way

- Covered all the essentials in Goldstein
  - Lagrangian, conservation laws, special relativity, Hamiltonian, canonical transformations
  - Central force, rigid body, oscillation
- Also talked about a lot of frivolous but intriguing topics

I don’t expect you to keep everything in your brain

- Hopefully, it will come back and help you when you see it in the more advanced courses of physics
  - At least you’ll know which book to look up