Lecture #13
Special Relativity

Purcell 5.3–5.9, 6.7

What We Did Last Time

Biot-Savart Law
\[ dA = \frac{IdL}{cr} \quad dB = \frac{IdL \times \hat{r}}{cr^2} \]

Special relativity recap
- Lorentz transformation, time dilation, length contraction
- Ladder and shed puzzle

I expect that you’ve reviewed your 15a/16 notes on
- Velocity addition: \[ u'_x = \frac{u_x - v}{1 - vu_x/c^2} \quad u'_y = \frac{u_y}{\gamma(1 - vu_x/c^2)} \]
- Momentum, energy
Today’s Goals

Study how Relativity and Electromagnetism work together
- How do E and B fields transform between two inertial frames?

Case study: a charged particle near a line of current
- Connection between electric and magnetic forces

Transformation of Forces

In frame O, a force \( \mathbf{F} = (F_x, F_y, 0) \) acts on an object of mass \( m \), initially at rest (\( \mathbf{p} = 0 \) at \( t = 0 \))
- Observe this for a very short time \( \Delta t \) in O

Force is defined by \( \mathbf{F} = d\mathbf{p}/dt \)
- Momentum grows as \( \Delta \mathbf{p} = \mathbf{F} \Delta t \)
- Position changes as \( \Delta \mathbf{x} = \frac{1}{2} \mathbf{a} (\Delta t)^2 = \frac{1}{2} m (\mathbf{F} \Delta t)^2 \)
- Energy increases as \( \Delta E = \frac{1}{2} \frac{(\Delta \mathbf{p})^2}{m} = \frac{1}{2} \frac{(F \Delta t)^2}{m} \)

Both \( \Delta \mathbf{x} \) and \( \Delta E \) are proportional to \( (\Delta t)^2 \)
- That means \( d\mathbf{x}/dt = 0 \), \( dE/dt = 0 \)
Transformation of Forces

Observe the same motion in $O'$

- For the $x'$ component

$$F'_x = \gamma \left( \frac{dp_x}{dt}' - \frac{v}{c^2} \frac{dx}{dt} \right) = \gamma \left( \frac{dp_x}{dt} - \frac{v}{c^2} \frac{dx}{dt} \right) - \frac{v}{c^2} \frac{dE}{dt} = 0$$

- For the $y'$ component

$$F'_y = \gamma \left( \frac{dp_y}{dt}' - \frac{v}{c^2} \frac{dx}{dt} \right) = \gamma \left( \frac{dp_y}{dt} - \frac{v}{c^2} \frac{dx}{dt} \right) - \frac{v}{c^2} \frac{dE}{dt} = 0$$

- Same for the $z'$ component

Forces transform as $F' = F_{||} + F_{\perp}/\gamma$

NB: $O$ is the rest frame of the object

Relativity of Electromagnetism

Ready to tackle E&M in moving frames?
We assume relativity:

The laws of electromagnetism (Maxwell and others) are the same in any inertial frame

We further assume:

Values of electric charges are the same in any inertial frame

- "Electric charge is a Lorentz scalar"
- Reasonable, but not obviously true

We will build our theory, and see where it leads us

- Is the theory self-consistent?
- Does the theory match experimental facts?
A charged particle is moving near an infinitely-long current

In the laboratory frame $O$, the wire contains:
- Positive ions: line density $\lambda_0$ esu/cm, velocity $0$
- Electrons: line density $-\lambda_0$ esu/cm, velocity $v_0$

Net current on wire: $I = \lambda_0 v_0$
Net charge on wire = 0
Therefore no electric field around the wire

Observe in the charge’s rest frame $O'$
- Positive ions are moving with velocity $-v$
- Length contraction increases line density to $\gamma \lambda_0$
- Electrons are moving with velocity $v_0'$ given by
  \[ v_0' = \frac{v_0 - v}{1 - \frac{v_0 v}{c^2}} \]
  or
  \[ \beta_0' = \frac{\beta_0 - \beta}{1 - \beta_0 \beta} \]
- The line density is?
Line Density of Electrons

Length contraction is relative to "natural" length at rest
- Electrons are not at rest in either $O$ or $O'$
- Line density $-\lambda_0$ esu/cm was given in the lab frame $O$
- The rest-frame density must be $\lambda_{\text{rest}} = -\frac{\lambda_0}{\gamma_0}$ where $\gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}}$
- Moving to the $O'$ frame, the density becomes
  \[-\frac{\lambda_0}{\gamma_0 \sqrt{1 - \beta_0^2}} = \frac{\lambda_0}{\gamma_0 (1 - \beta^2)} = -\lambda_0 \gamma (1 - \beta \beta_0)\]

Total line charge of the wire in the $O'$ frame is
\[\gamma \lambda_0 - \lambda_0 \gamma (1 - \beta \beta_0) = \gamma \beta \beta_0 \lambda_0\]
- The wire appears electrically charged from the moving particle

Force on the Charge

In $O'$
- Electric field due to wire charge is $E' = \frac{2\gamma \beta \beta_0 \lambda_0}{r'}$
- Repels if $v$ and $v_0$ are parallel

Going back to the $O$ (laboratory) frame
\[F = \frac{F'}{\gamma} = \frac{2q \beta \beta_0 \lambda_0}{r} = \frac{2qvl}{c^2 r}\] Note that $O'$ is the particle’s rest frame

- Particle receives a velocity-dependent force from the current
- Can rewrite as $F = \frac{q}{c} v \times \left(-\frac{2l}{cr} \hat{\phi}\right)$ Lorentz force and B field
What Happened?

Starting from
- Special Relativity,
- Lorentz invariance of the electric charge, and
- Coulomb's Law,

we have found a velocity-dependent force on a moving charge near a current
- This is the force we used earlier to define magnetic field

Magnetism is a natural consequence of electricity and special relativity

More generally

Electricity and magnetism are two manifestations of the same physics = Electromagnetism

Transformation of $E$, $B$ Fields

We have just seen that a $B$ field in one frame appears to be an $E$ field in another frame

There must be a set of transformation rules that can connect $(E, B)$ in $O$ to $(E', B')$ in $O'$
- Expect the rules to be linear, i.e., each component of $E'$, $B'$ is a linear function of the components of $E$, $B$
  - So that the superposition principle works in both frames
  - The rules must agree with the transformation of forces

I’ll show how $E$ transforms into $(E', B')$ first
- $B$ transforming into $(E', B')$ is similar, and is discussed in the textbook, so I may skip if time is short
E Field in Motion

Create uniform $E$ with parallel plates
- Plates are stationary in $O$, and parallel to the $x$-$z$ plane
- Charge density $+\sigma$ (and $-\sigma$) gives $E = 4\pi\sigma \hat{y}$

Look at this in $O'$
- Plates are moving with velocity $-v$
- Charge is unchanged
- But the plate length $L$ shrinks to $L' = L/\gamma$
- Charge density increases $\sigma' = \gamma\sigma$

$$E'_{\perp} = \gamma E_{\perp}$$

Electric field perp. to motion is increased by a factor $\gamma$

E Field $\rightarrow$ B Field

In $O'$ the charges are moving in $-x'$
- Two current sheets create a uniform $B$ field in between
- Current densities are $J' = \sigma'v = \gamma\sigma v$
- Careful with the directions of $J'$!

This produces uniform $B$ field between the current sheets in the $-z'$ direction

$$B'_{\perp} = -\frac{4\pi J'}{c} \hat{z} = -\frac{4\pi\gamma\sigma v}{c} \hat{z} = -\gamma\beta \times E_{\perp}$$

Electric field perp. to motion creates magnetic field
E Field in Motion

Now orient the plates in the $y$-$z$ plane
Boost again from $O$ to $O'$
- Distance between the plates shrink
- But the charge density remains same

$E' = E$

Electric field parallel to motion is unchanged

$E$ field parallel to motion does not create $B$
- $J$ is parallel to $x$-axis and uniform in $y$-$z$
- $A$ must be parallel to $x$-axis and uniform in $y$-$z$
- $\nabla \times A = 0$

Consistency

Combining what we found so far, we have

$E' = E + \gamma E_\perp$, \hspace{1em} $B' = -\gamma \beta \times E_\perp$

- Is this consistent with the force transformation?

Consider a charged particle at rest in the lab ($O$) frame
- In $O$, the force on the particle is $F = qE$
- In $O'$, the Lorentz force on the same particle is

$F' = qE - q\beta \times B' \leftarrow$ Velocity of the particle is $-v$ in $O'$

$= q(E_\parallel + \gamma E_\perp) + q\beta \times (\gamma \beta \times E_\perp)$

$= qE_\parallel + q\gamma E_\perp + q\gamma \left[ \beta (\beta \cdot E_\perp) - E_\perp (\beta \cdot \beta) \right]$

$= qE_\parallel + q\gamma (1 - \beta^2) E_\perp$

$= qE_\parallel + q E_\perp / \gamma = F_\parallel + F_\perp / \gamma$ Consistent with the force transformation
Boosting B Fields

Create uniform B field by a solenoid in O and observe it in O’

- Solenoid shrinks, so the winding density increases
  \[ n' = \gamma n \]

- How about the current?
  \[ I = \frac{\Delta Q}{\Delta t} \quad I' = \frac{\Delta Q'}{\Delta t'} = \frac{\Delta Q}{\gamma \Delta t} = \frac{I}{\gamma} \]

- The two effects cancel each other in B field
  \[ B'_{||} = B_{||} \]

Magnetic field parallel to motion is unchanged

Boosting B Fields

It’s harder to boost B field perpendicular to motion
We’ve already done boosting current on a straight wire

- In the lab frame O

  Positive ions: line density \( \lambda_0 \) esu/cm, velocity 0
  Electrons: line density \(-\lambda_0\) esu/cm, velocity \( v_0 \)

- In a moving frame O’

  Positive ions: line density \( \gamma \lambda_0 \) esu/cm, velocity \(-v\)
  Electrons: line density \(-\lambda_0 \gamma (1 - \beta v_0)\) esu/cm, velocity \( \frac{v_0 - v}{1 - \beta \beta_0} \)
**Boosting B Fields**

Net charge density \( \lambda \) and net current \( I \) on the wire are
- In the lab frame \( O \): \( \lambda = \lambda_0 - \lambda_0 \) \( I = \lambda_0 v_0 \)
- In the moving frame \( O' \):
  \[ \lambda' = \gamma \lambda_0 - \lambda_0 \gamma (1 - \beta \beta_0) = \gamma \beta \beta_0 \lambda_0 = \frac{\gamma I}{c} \]
  \[ I' = \gamma \lambda_0 \cdot v + \lambda_0 \gamma (1 - \beta \beta_0) \cdot \frac{v_0 - v}{1 - \beta \beta_0} = \gamma \lambda_0 v_0 = \gamma I \]

\( \mathbf{E} \) and \( \mathbf{B} \) fields at distance \( r \) from the wire are
- \( B_\perp = \frac{2I}{cr} \)
  \[ \begin{align*}
  B'_{\perp} &= \frac{2'I}{cr} = \frac{2\gamma I}{cr} = \gamma B_\perp \\
  E'_{\perp} &= \frac{2\lambda'}{r} = \frac{2\gamma \beta I}{cr} = \gamma \beta B_\perp
  \end{align*} \]

Magnetic field increases
Electric field is created

**Transformation of E, B Fields**

Complete set of field transformations:
- \( \mathbf{E}' = \mathbf{E} \parallel \quad \mathbf{E}' = \gamma (\mathbf{E}_\perp + \beta \times \mathbf{B}_\perp) \)
- \( \mathbf{B}' = \mathbf{B} \parallel \quad \mathbf{B}' = \gamma (\mathbf{B}_\perp - \beta \times \mathbf{E}_\perp) \)

- Striking symmetry between \( \mathbf{E} \) and \( \mathbf{B} \)
- Taking \( v \) in the +x direction as usual

\[ \begin{align*}
  E'_{x} &= E_{x} \\
  E'_{y} &= \gamma (E_{y} - \beta B_{z}) \\
  E'_{z} &= \gamma (E_{z} + \beta B_{y}) \\
  B'_{x} &= B_{x} \\
  B'_{y} &= \gamma (B_{y} + \beta E_{z}) \\
  B'_{z} &= \gamma (B_{z} - \beta E_{y})
  \end{align*} \]

Linear transformation, as expected

Note: \( \mathbf{E} \) and \( \mathbf{B} \) are not Lorentz four-vectors
- \((\phi, A/c)\) is — but that will be another lecture
Summary

Moving charge near current receives force proportional to its velocity
- Identified as Lorentz force due to magnetic field
- Electricity and magnetism are connected through relativity

Relativistic transformation of $E$ and $B$ fields

\[
\begin{align*}
E'_\parallel &= E_\parallel \\
E'_\perp &= \gamma (E_\perp + \beta \times B_\perp) \\
B'_\parallel &= B_\parallel \\
B'_\perp &= \gamma (B_\perp - \beta \times E_\perp)
\end{align*}
\]

- Linear transformation
- Consistent with the force transformation $F' = F_\parallel + F_\perp / \gamma$