What We Did Last Time

*LC circuit* is a simple harmonic oscillator
- \( L \sim \text{mass}, \ C \sim \text{spring} \)

*RLC circuit* is a damped oscillator
- Behavior depends on \( R \)
- Weak/strong/critical damping

AC voltage source

\[
V = V_0 \cos \omega t \quad \text{and} \quad \bar{P} = \frac{V_0 I_0}{2} = V_{\text{rms}} I_{\text{rms}}
\]
Today's Goals

Introduce Impedance
- Extension of resistance into AC circuits
- $R$, $L$, and $C$ (and combinations) can be treated together
- Catch: must do this in complex numbers

Use impedance to analyze simple AC circuits

1. Frequency filters
   - Combine $R$, $L$, or $C$ to selectively remove (= filter out) signals with high or low frequencies

2. Resonance circuit
   - Combine $R + L + C$ to enhance signals near a particular frequency
   - Quality Factor $Q$

Impedance

Impedance is a generalization of resistance
- Drive "something" with an AC voltage source
- Could be an $R$, $C$, $L$ or a combination

How does the current relate to the voltage?
- For $R$, the answer is simple:
  \[ V_0 \cos \omega t = R \cdot I_0 \cos \omega t \quad \Rightarrow \quad V_0 = R \cdot I_0 \]
- It would be nice if $L$ and $C$ were as simple as $V_0 = Z \cdot I_0$
- Not possible as long as we keep using $\cos \omega t$ and $\sin \omega t$

To simplify, we introduce a complex notation
\[ V(t) = \text{Re}(V_0 e^{i\omega t}) = V_0 \cos \omega t \]

Omit "Re" as implicitly being there
Impedance of C

Let’s study C first

\[ Q(t) = CV(t) = CV_0 e^{i\omega t} \]

\[ I(t) = \frac{dQ(t)}{dt} = CV_0 i\omega e^{i\omega t} \]

- The (complex) current can be written as
  \[ I(t) = I_0 e^{i\omega t} \] where \( I_0 = i\omega CV_0 \)
- If we take the real part
  \[ I(t) = -CV_0 \omega \sin \omega t \]

Define the **impedance of a capacitor** by

\[ V_0 = Z \cdot I_0 \]

\[ Z(\omega) = \frac{V_0}{I_0} = \frac{1}{i\omega C} \]

- This is a frequency-dependent imaginary number

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Impedance of L

For an L connected to the same source

\[ V(t) = V_0 e^{i\omega t} = L \frac{dI(t)}{dt} \]

\[ I(t) = \frac{V_0}{L} \int e^{i\omega t} dt = \frac{V_0}{i\omega L} e^{i\omega t} \]

- Therefore \( I_0 = \frac{V_0}{i\omega L} \)
- Take the real part \( I(t) = \frac{V_0}{L\omega} \sin \omega t \)

The **impedance of an inductor** is

\[ Z = \frac{V_0}{I_0} = i\omega L \]

- Another frequency-dependent imaginary number
Magnitude of Impedance

Impedance of $L$ and $C$ depends on the frequency

$$V_0 = Z(\omega)I_0$$

Impedance varies depending on the frequency $\omega$

$$Z_L = i\omega L$$

$$Z_R = R$$

$$Z_C = \frac{1}{i\omega C}$$

Phase of Impedance

Capacitor $C$

$$l(t) = \text{Re}(i\omega CV_0 e^{i\omega t})$$

$$= -\omega CV_0 \sin \omega t$$

- Voltage lags the current by $90^\circ$

Inductor $L$

$$l(t) = \text{Re}\left(\frac{V_0}{i\omega L} e^{i\omega t}\right)$$

$$= \frac{V_0}{\omega L} \sin \omega t$$

- Voltage leads the current by $90^\circ$

Both amplitudes and phases are expressed in the impedance $Z$

$$Z_C = \frac{1}{i\omega C} \rightarrow \arg(Z_C) = -\frac{\pi}{2}$$

$$Z_L = i\omega L \rightarrow \arg(Z_L) = \frac{\pi}{2}$$
Combining Impedance

Total impedance of a network of resistors, capacitors, and inductors can be calculated by adding up $Z_R$, $Z_C$, and $Z_L$.

- Same rules as resistor addition — just in complex numbers

$$Z = R + i\omega L$$

Impedance summarizes the network’s AC characteristics

$$Z = \frac{V(t)}{I(t)} = \frac{V_0 e^{i\omega t}}{I_0 e^{i\omega t}} = \frac{V_0}{I_0}$$

- Magnitude $|Z|$ is the ratio between peak voltage and peak current
- Complex phase $\text{arg}(Z)$ is the phase difference between voltage and current

Low Pass Filter

Series $R + C$ is driven by an AC voltage source

- Total impedance is $Z_{RC} = R + \frac{1}{i\omega C}$
- Current is
  $$I(t) = \frac{V_0 e^{i\omega t}}{Z_{RC}} = \frac{V_0}{R + 1/i\omega C} e^{i\omega t}$$
  $$V_{in} = V_0 e^{i\omega t}$$

The “output” voltage around $C$ is

$$V_{out}(t) = Z_C I(t) = \frac{1}{i\omega C} \frac{V_0}{R + 1/i\omega C} e^{i\omega t} = \frac{V_0}{i\omega RC + 1} e^{i\omega t}$$

- At low frequency ($\omega RC << 1$), $V_{out}$ is close to $V_{in}$
- At high frequency ($\omega RC >> 1$), $V_{out}$ approaches zero

This network passes low freq. and blocks high freq.

= a low-pass filter
Low Pass Filter

The frequency response of a simple low-pass filter is

\[ \frac{V_{out}}{V_{in}} = \frac{1}{i\omega RC + 1} = \frac{1}{\sqrt{(\omega RC)^2 + 1}} \]

- In log-log plot, it’s two straight lines joined by a round corner

The cut-off frequency is

\[ \omega = \frac{1}{RC} \quad f = \frac{1}{2\pi RC} \]

- This is where \( V_{out} = \frac{1}{\sqrt{2}} V_{in} \)

Above the cut-off, \( V_{out} \) drops inversely proportionally to \( \omega \)

- Such filter is called a first-order filter
- EE people call it a “−6 dB/octave” or “−20 dB/decade” filter

Other Filters

Low-pass and high-pass filters can be built with RC or RL

- All are first-order, ±6 dB/octave filters
- Steeper filters are usually made with active electronics
- Nowadays often with digital signal processing
Series RLC Circuit

An AC voltage source drives $R + L + C$
- Current is common $I = I_0 e^{i\omega t}$
- Total voltage is $V = V_0 e^{i\omega t} = V_R + V_L + V_C$

Use impedance

$$\frac{V_0 e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} I_0 e^{i\omega t}$$

Remember: addition of impedance works in the same way as resistance
- It just has to be done in the complex plane

Resonance

Total impedance changes with $\omega$
- $Z_L = i\omega L$ grows, $Z_C = 1/i\omega C$ shrinks

Magnitude is

$$|Z_{\text{total}}| = \left| R + i\omega L + \frac{1}{i\omega C} \right| = \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}$$

Minimum when $Z_L$ and $Z_C$ cancel each other

$$\min |Z_{\text{total}}| = R \quad \text{when} \quad i\omega L + \frac{1}{i\omega C} = 0 \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

At the resonance
- Impedance is minimum $\Rightarrow$ Current is maximum
- Impedance is real $\Rightarrow$ Current and voltage are in phase
Resonance

At the resonance
- Impedance is minimum ➔ Current is maximum
- Impedance is real ➔ Current and voltage are in phase

\[
\left| \frac{I_0}{V_0} \right| = \left| Z \right| = \frac{1}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}
\]

Current peaks at the resonance
Peak is higher and narrower for smaller \( R \)

Phase

Phase of impedance tells us if \( V \) and \( I \) are synchronized

\[
\arg(Z) = \arctan\left( \frac{\omega L - \frac{1}{\omega C}}{R} \right)
\]

- It changes from \(-90^\circ\) to \(+90^\circ\) as \( \omega \) crosses the resonance

Current leads voltage below the resonance
Current lags behind voltage above the resonance

Phase shift is more abrupt for smaller \( R \)
Quality Factor

How steep (or narrow) is the resonance?
- Look at the frequencies where \( \text{Im}(Z) \) and \( \text{Re}(Z) \) are the same size
- That’s where \( I_0 \) is \( 1/\sqrt{2} \) times the peak value
- Suppose the damping is weak, i.e., \( R \ll 2\sqrt{L/C} \)
  
\[ \omega^2 = \frac{1}{LC} \pm \frac{R}{L} \sqrt{\frac{1}{LC} - \omega_0^2} \]
- Difference between the two solutions \( \omega_1 \) and \( \omega_2 \) \(( \omega_1 > \omega_2 )\) is
  
\[ \frac{\omega_1 - \omega_2}{\omega_0} = \frac{R}{\omega_0 L} = \frac{1}{Q} \]

The larger the \( Q \), the narrower and steeper the resonance

What is \( Q \)?

\( Q \) is the relative width (in frequency) of the resonance
- Ex: FM stations use 87.5–108 MHz signals separated by 200 kHz
- The radio’s tuning circuit must have \( Q > Q_{\text{min}} = \frac{108 \times 10^6 \text{Hz}}{200 \times 10^3 \text{Hz}} = 504 \)

Think of the same \( RLC \) circuit as a weak-damped oscillator
- The solution was \( Q(t) = e^{-\alpha t} e^{\pm i\omega t} \)
- The decay time is \( \tau = \frac{1}{\alpha} = \frac{2L}{R} \)
- In that time, it oscillates by
  
\[ \omega \tau = \omega_0 \frac{2L}{R} = 2Q \text{ radian} \]

\( Q \) is (proportional to) the number of oscillations it makes before dying
Summary

Impedance of $R$, $C$, and $L$

$$Z_R = \frac{V_R}{I_R} = R \quad Z_C = \frac{V_C}{I_C} = \frac{1}{i\omega C} \quad Z_L = \frac{V_L}{I_L} = i\omega L$$

- Generally a frequency-dependent complex number

Low- and high-pass filters

- Cut-off frequency $\omega_{\text{cutoff}} = \frac{1}{RC}$ or $\omega_{\text{cutoff}} = \frac{R}{L}$

Resonance of an RLC circuit $\omega_0 = \frac{1}{\sqrt{LC}}$

- Current amplitude peaks
- Phase between voltage and current changes by 180°

Quality factor $Q$

- $Q = \frac{\omega_0 L}{R}$
  
  = narrowsness (steepness) of the resonance
  
  = how many "rings" a weakly damped oscillators make