What We Did Last Time

**Impedance** of $R$, $C$, and $L$

$$Z_R = \frac{V_R}{I_R} = R \quad Z_C = \frac{V_C}{I_C} = \frac{1}{i\omega C} \quad Z_L = \frac{V_L}{I_L} = i\omega L$$

- Generally a frequency-dependent complex number

**Low- and high-pass filters**
- Cut-off frequency $\omega_{\text{cutoff}} = \frac{1}{RC}$ or $\omega_{\text{cutoff}} = \frac{R}{L}$

**Resonance** of an RLC circuit $\omega_0 = \frac{1}{\sqrt{LC}}$
- Current amplitude peaks
- Phase between voltage and current changes by 180°

**Quality factor $Q$**
- $Q = \frac{\omega_0 L}{R}$
  - = narrowness (steepness) of the resonance
Today’s Goals

Introduce displacement current
- The last element of Maxwell’s equations

Complete Maxwell’s equations

Study Maxwell’s eqns. in vacuum
- Derive wave equations
- Find a solution
  → Electromagnetic waves

Incomplete Equations

In Lecture #14, we got this set of equations

\[
\begin{align*}
\nabla \cdot E &= 4\pi \rho \\
\nabla \cdot B &= 0 \\
\n\nabla \times E &= -\frac{1}{c} \frac{\partial B}{\partial t} \\
\n\nabla \times B &= \frac{4\pi}{c} J
\end{align*}
\]

There is a small problem
- \( \nabla \times B = \frac{4\pi}{c} J \) means \( \text{div } J \) must be zero
- Charge conservation tells us \( \text{div } J = -\frac{\partial \rho}{\partial t} \)
- \( \frac{\partial \rho}{\partial t} \) may not be zero if the system is time-dependent

The above equations work only for stationary charge distributions
Fixing the Inconsistency

Something must be done to \( \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \)
- Try adding a vector \( \mathbf{F} \) to the rhs
- What is \( \mathbf{F} \)? Remember that div of the lhs is 0

\[
\nabla \cdot \left( \frac{4\pi}{c} \mathbf{J} + \mathbf{F} \right) = 0 \rightarrow \nabla \cdot \mathbf{F} = -\frac{4\pi}{c} \nabla \cdot \mathbf{J} = \frac{4\pi}{c} \frac{\partial \rho}{\partial t}
\]
- Take time derivative of Gauss’s law

\[
\nabla \cdot \mathbf{E} = 4\pi \rho \quad \text{time deriv.} \quad \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = 4\pi \frac{\partial \rho}{\partial t}
\]
- Compare the rhs

\[
\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = \nabla \cdot \mathbf{F} \rightarrow \mathbf{F} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
\]
We’ve found the missing piece!

Displacement Current

New-and-improved Ampère’s Law:
\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
\]
Second term can be seen as an additional “current”

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \left( \mathbf{J} + \mathbf{J}_d \right) \quad \text{where} \quad \mathbf{J}_d \equiv \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}
\]
- The (obscure) name is historical

Displacement current \( \mathbf{J}_d \) is not a real current
- It does not describe charges flowing through some region
- but it acts like a real current
- Let’s see how it fits in an example
Displacement Current

Consider a charging capacitor

\[ l = \frac{dQ}{dt} \]

Apply Ampère to loop \( C \) \( \Rightarrow \) \( \oint_C \mathbf{B} \cdot d\mathbf{s} = \frac{4\pi}{c} l = \frac{4\pi}{c} \int_S \mathbf{J} \cdot d\mathbf{a} \)

- Fine, but this is supposed to hold for any \( S \) that's bounded by \( C \)

Choose \( S' \) that intersects the capacitor \( \Rightarrow \) \( \mathbf{J} = 0 \) on \( S' \)

- Naive Ampère fails because \( \int_{S'} \mathbf{J} \cdot d\mathbf{a} = 0 \)

Displacement Current

\( \mathbf{E} \) field between the plates is increasing with time

- If the capacitor has an area \( A \), \( \mathbf{E} = \frac{4\pi Q}{A} \) \( \Rightarrow \) \( \frac{d\mathbf{E}}{dt} = \frac{4\pi}{A} \frac{dQ}{dt} = \frac{4\pi}{A} l \)

Displacement current is

\( \mathbf{J}_d = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} = \frac{l}{A} \Rightarrow \int_{S'} \mathbf{J}_d \cdot d\mathbf{a} = \int_{S'} \frac{l}{A} d\mathbf{a} = l \)

“Extended” Ampère works \( \oint_C \mathbf{B} \cdot d\mathbf{s} = \frac{4\pi}{c} \int_S (\mathbf{J} + \mathbf{J}_d) \cdot d\mathbf{a} = \frac{4\pi}{c} l \)
More Generally

For Ampère’s Law to work consistently, \(\frac{4\pi}{c} \int_S (\mathbf{J} + J_d) \cdot d\mathbf{a}\) must depend only on the border of \(S\)

- Using charge conservation
  \[\int_S \mathbf{J} \cdot d\mathbf{a} - \int_{S'} \mathbf{J} \cdot d\mathbf{a}' = -\frac{dQ_i}{dt}\]

- Gauss’s Law tells us
  \[\int_S \mathbf{E} \cdot d\mathbf{a} - \int_{S'} \mathbf{E} \cdot d\mathbf{a}' = 4\pi Q_i\]

\[\int_S \mathbf{J} \cdot d\mathbf{a} - \int_{S'} \mathbf{J} \cdot d\mathbf{a}' = -\frac{1}{4\pi} \frac{d}{dt} \left(\int_S \mathbf{E} \cdot d\mathbf{a} - \int_{S'} \mathbf{E} \cdot d\mathbf{a}'\right)\]

\[\int_S \left(\mathbf{J} + \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a} = \int_{S'} \left(\mathbf{J} + \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a}'\]

Complete Maxwell’s Equations

In differential forms:

**CGS**

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 4\pi \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

**SI**

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

- Maxwell introduced the last term (in 1861) based purely on the argument of mathematical consistency

**Typo in Purcell §9.3 (15')**
Integral Forms

First two equations apply to any volume $V$ enclosed by surface $S$
Second two apply to any surface $S$ bounded by a contour $C$
$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{c} \frac{d\Phi_B}{dt}$
$\oint_C \mathbf{B} \cdot d\mathbf{s} = \frac{4\pi}{c} I + \frac{1}{c} \frac{d\Phi_E}{dt}$

Maxwell in Vacuum

In vacuum, where $\rho = 0$ and $J = 0$,
$$\nabla \cdot \mathbf{E} = 4\pi \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

No source $\Rightarrow$ No field?
- If $\mathbf{E}$ and $\mathbf{B}$ are both changing with time, they can create each other
- It’s like pulling yourself up with bootstraps?

Let’s try to solve these equations together

CGS

$\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$
$\Phi_B \equiv \int_S \mathbf{B} \cdot d\mathbf{a}$
$Q \equiv \int_V \rho \, dv$
$I \equiv \int_S \mathbf{J} \cdot d\mathbf{a}$
Solving Maxwell in Vacuum

Strategy: Decouple $E$ and $M$

Faraday $\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}$

Same except for $-1/c$ $\nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t}$

$\nabla \times (\nabla \times E) = \nabla \times \left(-\frac{1}{c} \frac{\partial B}{\partial t}\right)$

$\nabla \times (\nabla \times B) = \frac{\partial}{\partial t} \left(\nabla \times B\right) = \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial E}{\partial t}\right)$

Faraday Ampère

$\nabla \times (\nabla \times E) = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$

$\nabla \times (\nabla \times B) = \frac{\partial^2 B}{\partial t^2}$

$\nabla \times (\nabla \times E) = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$

An $E$-only differential equation

How about $B$?

Solving Maxwell in Vacuum

Repeat, but this time try to eliminate $E$

Ampère $\nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t}$

Same except for $1/c$ $\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}$

$\nabla \times (\nabla \times B) = \nabla \times \left(\frac{1}{c} \frac{\partial E}{\partial t}\right)$

$\nabla \times (\nabla \times E) = \frac{\partial}{\partial t} \left(\nabla \times E\right) = \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial B}{\partial t}\right)$

$\nabla \times (\nabla \times B) = -\frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}$

$\nabla \times (\nabla \times E) = \frac{\partial^2 E}{\partial t^2}$

Always 0

$\nabla \times (\nabla \times B) = 0$

$\nabla \times (\nabla \times E) = 0$

A $B$-only differential equation

Now we must solve $\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$ and $\nabla^2 B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}$
1-D Wave Solutions

Suppose $E(x,y,z,t) = E(x,t)$, i.e. no $y$ and $z$ dependence.

\[
\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial x^2}
\]

This can be satisfied if $E(x,t) = f(x \pm ct)$

\[
\text{lhs} = f''(x \pm ct) \quad \Rightarrow \quad \text{rhs} = \frac{1}{c^2} (\pm c)^2 f''(x \pm ct)
\]

What is $f(x \pm ct)$?

- At $t = 0$, $f(x)$ is an arbitrary vector function of $x$
- As $t$ increases, $f(x \pm ct)$ moves along the $x$ axis
  - $f(x + ct)$ moves toward negative $x$ with velocity $-c$
  - $f(x - ct)$ moves toward positive $x$ with velocity $+c$

Waves propagating with the speed of light

Electromagnetic Waves

Solutions to Maxwell’s equations in vacuum are waves of $E$ and $B$ fields = electromagnetic waves

- Maxwell, based on the experimental data of the day, found the speed was $3.1 \times 10^8$ m/s

  “We can scarcely avoid the conclusion that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena”

- Electricity and magnetism were unified with optics

In CGS, the speed comes out to be $c$

- This is because of the $1/c$ in Faraday and Ampère
- We checked that $1/c$ came naturally out of Special Relativity
  - NB: we only used Coulomb + Relativity there
Summary

Complete Maxwell’s equations

\[ \nabla \cdot \mathbf{E} = 4\pi \rho \quad \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \] 

- Displacement current \( \mathbf{J}_d = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \) needed for mathematical consistency

Electromagnetic waves

- Maxwell’s eqns. in vacuum \( \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \)

- Solutions are waves propagating with speed of light
  ... which is light itself