What We Did Last Time

Complete Maxwell’s equations

\[ \nabla \cdot \mathbf{E} = 4\pi \rho \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

- Displacement current \( \mathbf{J}_d = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \) needed for mathematical consistency

Electromagnetic waves

- Maxwell’s eqns. in vacuum \( \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \)

- Solutions are waves propagating with speed of light

... which is light itself
Today’s Goals

Examine electromagnetic waves in more details
- Plane waves and their completeness
- Wave number \( k \) and angular frequency \( \omega \)
- Relationship between \( E \) and \( B \)
- Polarization

Consider the energy carried by electromagnetic waves
- Poynting vector
- Power density and momentum density

Sinusoidal Wave Solution

Let’s consider sinusoidal \( E \) waves traveling along \( +x \)
\[ E = E_0 \sin(kx - \omega t) \]
- \( k \): wave number [unit = radian/cm]
  is related to the wave length \( \lambda \) by \( k\lambda = 2\pi \) \[ k = \frac{2\pi}{\lambda} \]
- Just as \( \omega \) [unit = radian/s] is related to the period \( T \) by \( \omega T = 2\pi \)

In order to satisfy the wave equation \( \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \)
\[ \frac{\partial^2}{\partial x^2} \sin(kx - \omega t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \sin(kx - \omega t) \]
\[ k^2 = \frac{\omega^2}{c^2} \] \[ \omega = ck \]
- With this condition, \( \sin(kx - \omega t) = \sin k(x - ct) \)

Expected from general soln. \( f(x \pm ct) \)
Sinusoidal Wave Solution

There are more equations to satisfy
\[ \nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{E}_0 \sin(kx - \omega t) = \hat{x} \cdot \mathbf{E}_0 k \cos(kx - \omega t) = 0 \]
- \( \mathbf{E} \) is perpendicular to \( \mathbf{x} \), the direction of wave propagation
- Let's take the direction of \( \mathbf{E}_0 \) as the \( y \) axis \( \mathbf{E} = \hat{y}\mathbf{E}_0 \sin(kx - \omega t) \)

The last equation to satisfy
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \hat{y}\mathbf{E}_0 \sin(kx - \omega t) = -\hat{z}\mathbf{E}_0 ck \cos(kx - \omega t) \]
- Integrating by \( t \), we find \( \mathbf{B} = \hat{z}\mathbf{E}_0 \sin(kx - \omega t) \)
- \( \mathbf{B} \) is perpendicular to both \( \mathbf{E} \) and \( \mathbf{x} \), and \( |\mathbf{B}| = |\mathbf{E}| \)

Plane Waves

Plane sinusoidal EM waves are described by 3 vectors:
- \( \mathbf{E}_0 \): amplitude and direction of \( \mathbf{E} \) field
- \( \mathbf{B}_0 \): amplitude and direction of \( \mathbf{B} \) field
- \( \mathbf{k} \): wave number and direction of propagation

\[ \begin{align*}
\mathbf{k} &= k \hat{x} \\
\mathbf{E}_0 &= \mathbf{E}_0 \hat{y} \\
\mathbf{B}_0 &= \mathbf{E}_0 \hat{z}
\end{align*} \]

Generally
- \( \mathbf{E}_0, \mathbf{B}_0 \) and \( \mathbf{k} \) are perpendicular and right-handed
- \( |\mathbf{E}_0| = |\mathbf{B}_0| \)
General Plane Waves

General form of plane EM waves are

\[
\begin{align*}
E &= E_0 \sin(k \cdot r - \omega t) = E_0 \sin(k_x x + k_y y + k_z z - \omega t) \\
B &= B_0 \sin(k \cdot r - \omega t) = B_0 \sin(k_x x + k_y y + k_z z - \omega t)
\end{align*}
\]

with conditions \( \omega = kc, \quad |E_0| = |B_0|, \quad \hat{k} = \hat{E} \times \hat{B} \)

- ... plus the corresponding cosine terms

Fourier transform:

- Any (repetitive) function of space can be represented by an infinite sum of sines and cosines of the form

\[
\sum_k \left[ A_k \sin(k \cdot r) + B_k \cos(k \cdot r) \right]
\]

- Understanding plane waves is understanding all possible waves

Polarization

For a given \( k \), there are two possible directions of \( E \)

- \( E \) is in the \( y \)- or \( z \)-plane
- Direction of \( E \) is called the polarization

They are 2 independent solutions of the wave eqn

- Linear combinations make all the possibilities

Polarizing filters (a.k.a. Polaroids) can selectively absorb one polarization

- Used in photography, sun glasses, LCD, etc.
Energy

We know **light carries energy** with it
- Sun light brings (or brought) all the energy we use through vacuum between Sun and Earth

It contains nothing but **E** and **B** fields
- Energy density [erg/cm$^3$] of **E** and **B** fields are
- For plane waves
  \[
  \begin{align*}
  \mathbf{E} &= \hat{y}E_0 \sin(kx - \omega t) \\
  \mathbf{B} &= \hat{z}E_0 \sin(kx - \omega t)
  \end{align*}
  \]
  \[
  u = \frac{E_0^2}{4\pi} \sin^2(kx - \omega t)
  \]

This energy is moving with speed $c$

How much power arrives in a unit area?

Poynting Vector

Consider the energy $U = \frac{1}{8\pi} \int_V (E^2 + B^2) \, dv$ inside a volume $V$
- Time derivative
  \[
  \frac{dU}{dt} = \frac{1}{8\pi} \int_V \left(2 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} + 2 \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B}\right) \, dv
  \]
- Use Maxwell
  \[
  = \frac{1}{4\pi} \int_V \left(c \left(\nabla \times \mathbf{B}\right) \cdot \mathbf{E} - c \left(\nabla \times \mathbf{E}\right) \cdot \mathbf{B}\right) \, dv
  \]
  \[
  = -\frac{c}{4\pi} \int_V \nabla \cdot (\mathbf{E} \times \mathbf{B}) \, dv
  \]
- Divergence theor. $\quad -\int_S \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{B}\right) \cdot d\mathbf{a}$

This much energy leaks out from the surface $S$

Define Poynting vector $\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$
- $\mathbf{S}$ represents the amount and direction of energy flow per unit area
Poynting Vector

Poynting vector \( \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \) represents the flux of energy

- Per unit area and unit time
- It points in the direction of the EM energy flow

Check the dimension

\[
\begin{bmatrix}
\mathbf{S} \\
\mathbf{c}
\end{bmatrix} = \frac{c}{4\pi} \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \text{length} \\ \text{time} \end{bmatrix} \begin{bmatrix} \text{energy} \\ \text{volume} \end{bmatrix} = \begin{bmatrix} \text{length} \\ \text{energy} \end{bmatrix} \begin{bmatrix} \text{time} \\ \text{volume} \end{bmatrix} = \frac{\text{energy}}{\text{time} \cdot \text{volume}} = \frac{\text{power}}{\text{area}}
\]

- Unit in CGS: \( \text{erg/sec} \cdot \text{cm}^2 \)
- In SI, \( \mathbf{S} = \mathbf{E} \times \mathbf{B} \) has the unit of \( \text{watts/m}^2 \)

Magnitude of \( \mathbf{S} \) is also known as the intensity

\[
\mathbf{S} = c \mathbf{E} \times \mathbf{B} = c \mathbf{E} \mathbf{B}
\]

Back to Plane Waves

Consider a linearly polarized plane wave: \( \mathbf{E} = \hat{y} \mathbf{E}_0 \sin(kx - \omega t) \), \( \mathbf{B} = \hat{z} \mathbf{E}_0 \sin(kx - \omega t) \)

Poynting vector is

\[
\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \hat{x} \frac{c}{4\pi} \mathbf{E}_0^2 \sin^2(kx - \omega t)
\]

- Compare with the energy density \( u = \frac{\mathbf{E}_0^2}{4\pi} \sin^2(kx - \omega t) \)

\[
\mathbf{S} = cu \mathbf{k} = cu
\]

- This makes sense: energy distribution is moving with velocity \( c \)

For most interesting cases, the oscillation is very fast

- Visible light is \( \sim 10^{14} \) Hz

- What matters is the average power density and average intensity

\[
\bar{\mathbf{S}} = \frac{c}{4\pi} \mathbf{E}_0^2 \mathbf{k} = \frac{\mathbf{E}_0^2}{8\pi} \mathbf{k} \quad \bar{I} = \frac{c}{8\pi} \mathbf{E}_0^2
\]

mean square \( \mathbf{E} \) field
Charging Capacitor

Poynting vector appears whenever both E and B are there

- Consider a charging capacitor
  \[ E = -\frac{4\pi Q}{a^2} \hat{z} = -\frac{4Q}{a^2} \hat{z} \quad \frac{\partial E}{\partial t} = -\frac{4I}{a^2} \hat{z} \]

- "Extended" Ampère’s law gives
  \[ B = -\frac{2Ir}{ca^2} \hat{\phi} \]

- Poynting vector is
  \[ S = \frac{c}{4\pi} \left( -\frac{4Q}{a^2} \hat{z} \right) \left( -\frac{2Ir}{ca^2} \hat{\phi} \right) = -\frac{2QIr}{\pi a^4} \hat{r} \]

- Integrating over the cylindrical "side" surface
  \[ \int_{\text{side}} S \cdot da = \frac{2Qla}{\pi a^4} 2\pi ad = -\frac{QI}{a^2} \]

Energy flows inward

\[ \Rightarrow \text{Capacitor accumulates energy} \]

Momentum in EM Waves

EM waves carry not only energy but also momentum

- In relativity, E and p are related by
  \[ E^2 = |p|^2 c^2 + m^2 c^4 \]

- For EM radiation, \( m = 0 \)
  \[ E^2 = |p|^2 c^2 \quad |p| = \frac{E}{c} \]

This is energy, not E field!

- Relation \( E = pv \) is a general feature of all waves \( \Rightarrow \) Physics 15c

Energy flow is given by the Poynting vector

\[ \frac{\text{energy}}{\text{time} \cdot \text{area}} = \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \quad \frac{\text{momentum}}{\text{time} \cdot \text{area}} = \mathbf{S} = \frac{1}{4\pi} \frac{E}{c} \mathbf{E} \times \mathbf{B} \]

- Momentum / time is force; force / area is pressure

- EM waves exert a radiation pressure
  \[ \frac{S}{c} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B} \]
Radiation Pressure

Pressure due to EM radiation was predicted by Maxwell
- It was experimentally demonstrated 30 years later

Because $p = E/c$, the pressure is very small
- A 100 megawatt laser would produce a force
  $$F = \frac{P}{c} = \frac{10^8 \text{ watts}}{3 \times 10^8 \text{ m/s}} = 0.3 \text{ newtons}$$
  That’s an ounce in weight

It is important in extreme high temperatures
- Very large stars are supported by the radiation pressure from collapsing due to gravity

Summary

Plane wave solutions of Maxwell’s equations
$$\begin{align*}
\mathbf{E} &= E_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \\
\mathbf{B} &= B_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)
\end{align*}$$
- $\omega = kc$,
- $|\mathbf{E}_0| = |\mathbf{B}_0|$, $\hat{\mathbf{k}} = \hat{\mathbf{E}} \times \hat{\mathbf{B}}$
- Propagates along $\mathbf{k}$ with speed $c$
- Two possible polarizations for the same $\mathbf{k}$

Energy flow given by the Poynting vector $\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$
- For plane waves, average energy flow is $\mathbf{\bar{S}} = \frac{c}{4\pi} \mathbf{\bar{E}}^2 \hat{\mathbf{k}} = \frac{c}{8\pi} E_0^2 \hat{\mathbf{k}}$

EM waves also carry momentum $p = E/c$
- Radiation pressure is $\mathbf{S}/c$