Electromagnetism
Physics 15b

Lecture #2
Guass’s Law
Electric Field and Flux

Purcell 1.7–1.15

Administrativa

Online sectioning due Wednesday (Study Card Day)
- Go to http://www.section.fas.harvard.edu/
- Do both discussion sections and lab sections
- Email Giovanni zevi@physics.harvard.edu for questions

Participate in Learning Gains Study by Prof. Mazur
- Take two online questionnaires and get 2 extra points toward your course grade
  - CSEM = Conceptual Survey of Electricity and Magnetism
  - Once each at the beginning and end of the semester
- Go to http://galileo.harvard.edu/15b.html
  - You will have to register first by clicking “Not registered? Email”
- Email Jessica Watkins watkinsj@seas.harvard.edu or Doug van Wieren dvw@deas.harvard.edu for questions
What We Did Last Time

Electric charge = Source and recipient of electric forces
- Positive and negative; conserved; and quantized

Coulomb’s Law \( \mathbf{F}_{2} = \frac{q_{1}q_{2}}{r_{21}^{2}} \mathbf{r}_{21} \)
- Inverse-square law same as gravity
Multiple charges \( \implies \) Superposition Principle \( \mathbf{F}_{j} = \sum_{k \neq j} \mathbf{F}_{jk} \)

Electrostatic force is conservative
- Electrostatic energy depends only on the positions of the charges
- For 2-body:

\[
U = \frac{q_{1}q_{2}}{r_{12}}
\]
For N-body:

\[
U = \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} \frac{q_{j}q_{k}}{r_{jk}}
\]

Today’s Goals

Introduce electric field \( \mathbf{E} \)
- First (and the easiest) of the E&M fields
- Representation by field lines

Define flux \( \Phi \)
- How much \( \mathbf{E} \) flows through an area

Introduce Gauss’s Law
- Useful tool for many E&M problems

Apply Gauss’s Law to a few examples
- Spherical charge distribution
- Infinite sheet of charge

Discuss energy in the electric field
- … if I get that far
Electric Field

A charge $q_0$ is placed near $\{q_1, q_2, \ldots, q_N\}$

- Force on $q_0$: $F = \sum_{j=1}^{N} \frac{q_0 q_j}{r_{0j}^2} \hat{r}_{0j} = q_0 \sum_{j=1}^{N} \frac{q_j \hat{r}_{0j}}{r_{0j}^2}$

- Most of the “interesting” part has nothing to do with the value of $q_0$
- But the position of $q_0$ is important – Let’s call it $r$

$$r_{0j} = r - r_j \quad r_{0j} = |r - r_j| \quad \hat{r}_{0j} = \frac{r - r_j}{|r - r_j|}$$

$$F = q_0 \sum_{j} \frac{q_j (r - r_j)}{|r - r_j|^3}$$

A function of $r$ for a given configuration of $\{q_1, q_2, \ldots, q_N\}$

Imagine we have a fixed configuration of $\{q_1, q_2, \ldots, q_N\}$ and we can move $q_0$ around to “test” what happens to it

$$F(r) = q_0 \sum_{j} \frac{q_j (r - r_j)}{|r - r_j|^3}$$

$E(r)$ is the electric field
- It is a “vector field”, i.e. a vector function of the position $r$

We’ve introduced a middle-man for the electric force
- Charge creates $E$ field $\Rightarrow$ $E$ field creates force on charge
- End result is unchanged (of course)

Dimension of $E$ is force/charge
Unit of $E$ is dyne/esi
Electric Field

The electric field \( E \) created by a single charge \( q \) at the origin is

\[
E(r) = \frac{q}{r^2} \hat{r}
\]

This equation shows that the electric field points outward.

One can express the electric field graphically by small arrows. But it's busy and tedious to draw.

Faraday came up with an easier way to draw the electric field: field lines. Long, continuous arrows originating from the charge represent the direction.

What about the size of the electric field? 

Field Line Density

The electric field weakens with distance according to the inverse-square law:

\[
|E| = \frac{q}{r^2}
\]

Field lines become less crowded with distance.

- Consider a sphere at distance \( r \).
  - Surface area \( S = 4\pi r^2 \)
  - Number of field lines \( N \) is constant

\[
\text{Density} = \frac{N}{S} = \frac{N}{4\pi r^2}
\]

The density of field lines is proportional to the electric field magnitude if we choose the number of lines \( N \) to be proportional to \( q \).
Field Line Rules

You can often draw field lines without calculation

- Just follow a few rules

  - Field lines start from a positive charge and end in a negative charge
    - Exception: it’s OK to come from/go to infinitely far away
  - Field lines cannot split, merge, or cross each other
  - The number of field lines coming out of a charge is proportional to the amount of the charge

- The last rule ensures proportionality between the field line density and the electric field

Practice #1

+2

-1
Charge Distribution

Electric field created by multiple charges is

\[ \mathbf{E} = \sum_i \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \]

Superposition Principle

Electric charge may be distributed over objects, e.g.
- Along a length of wire (1-d)
- On the surface of a plate (2-d)
- Inside the volume of a ball (3-d)

For continuous charge distributions, we replace the sum with an integral

\[ \mathbf{E} = \int \frac{dq}{r^2} \hat{\mathbf{r}} \]

Integral over line/surface/volume
Uniform Linear Charge

Charge $Q$ uniformly distributed on a thin rod of length $\ell$

- What is the electric field $E$ at point $A$, which is distance $r$ away from the center of the rod?

Set up is everything

1. Define $x$-$y$ coordinates
2. Pick a small piece of the rod and call it $dx$
3. Calculate the field due to this piece
4. Integrate along the rod

The charge on the $dx$ piece is

Linear charge density (esu/cm)

- Field from this piece is $dE = \frac{Q}{\ell} \frac{dx}{r^2 + x^2}$

Note that the field from the green piece is same size and flipped in $x$

- Only the $y$-component will remain after adding up

$$dE_y = \frac{Q}{\ell} \frac{dx}{r^2 + x^2} \frac{r}{\sqrt{r^2 + x^2}}$$

- Integrate this!

$$E = \int_{-\ell/2}^{\ell/2} \frac{Q}{\ell} \frac{r}{(r^2 + x^2)^{3/2}} dx \hat{y}$$

$$= \frac{Q}{r \sqrt{r^2 + \ell^2/4}} \hat{y}$$

- Check the direction, dimension, $r$-dependence, large-$r$ limit
Concept of Flux

Consider a flow of water
- The water velocity is described by
  \[ \mathbf{v}(x, y, z) = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \equiv (v_x, v_y, v_z) \]

Immerse a tiny wire loop
- Area of the loop is \( a \)
- Define the loop area vector \( \mathbf{a} \) as being perpendicular to the loop, and the magnitude \( a \) equals the area of the loop
  - It represents the size and the orientation of the loop
  - If the loop is small, the shape is irrelevant

Q: how much water will flow through the loop?

\[ \mathbf{v} \cdot \mathbf{a} \quad \text{Generalize for a loop of not-so-small size} \]

\[ \int \mathbf{v} \cdot d\mathbf{a} \quad \text{Call this “water flux” through the loop} \]

Field Flux

Imagine a surface \( S \) in an electric field \( \mathbf{E} \)
- It could be any shape, any size, any angle

We can define the “flux” of the electric field through \( S \)

\[ \Phi = \int \mathbf{E} \cdot d\mathbf{a} \quad S \]

Density of the field lines is proportional to \( E \)
- So \( \Phi \) represents “how many field lines goes through \( S \)”
Electric Flux

Unit of $\Phi$ is $(\text{dyne/esu}) \times \text{cm}^2$
- This equals to esu, because $\text{esu} = \sqrt{\text{dyne} \cdot \text{cm}}$
- i.e., same as the unit of charge (???)

Sign of flux depends on the direction of $da$
- That is, you must define which side of $S$ is “positive”

Which Way is $da$?

Defined unambiguously only for a closed surface
- i.e., a surface that wraps around a volume completely
- At any point on the surface, $da$ is perpendicular to the surface
- By convention, $da$ points outward from of the surface

In other words: flux is positive (negative) if the net flow is coming out of (going into) the volume
Flux Through a Sphere

Consider a sphere of radius $r$ around a charge $q$

- $da$ always points outward, i.e. parallel to $r$

- Coulomb: $E = \frac{q}{r^2} \hat{r}$

\[
\Phi = \int_{\text{sphere}} E \cdot da = \int_{\text{sphere}} \left( \frac{q}{r^2} \hat{r} \right) \cdot (da \hat{r})
\]

\[
= \frac{q}{r^2} 4\pi r^2 = 4\pi q
\]

This is hardly surprising

- $\Phi$ should be proportional to the number of field lines coming out of the charge, which is proportional to $q$
- We just didn’t know that the constant was $4\pi$

Non-Spherical Surface

Now take an arbitrary closed surface $S$ around charge $q$

- Define $da$ at $r$ from the charge $q$
- Infinitesimal flux $d\Phi$ through $da$ is

\[
d\Phi = E \cdot da = \frac{q}{r^2} \cos \theta da
\]

- Suppose $da$ covers solid angle $d\Omega$ around $q$

\[
\cos \theta da = r^2 d\Omega
\]

- Integrate

\[
\Phi = \int_{S} E \cdot da = \int \frac{q}{4\pi r^2} r^2 d\Omega = q \int d\Omega = 4\pi q
\]

The flux does not depend on the shape of the surface
Gauss’s Law

Now consider multiple charges contained inside surface $S$
- Total $E$ field is a sum of fields from each charge
  \[
  \Phi = \int_{S} \left( \sum_{j} E_{j} \right) \cdot d\mathbf{a} = \sum_{j} \int_{S} E_{j} \cdot d\mathbf{a} = \sum_{j} 4\pi q_{j} = 4\pi \sum_{j} q_{j}
  \]

Net flux through a closed surface is given by the net charge inside the surface by
  \[
  \Phi = \int E \cdot d\mathbf{a} = 4\pi q_{\text{inside}}
  \]

The law connects charge and field in yet another way
- Coulomb’s law did it one way – they are equivalent

Spherical Charge Distribution

Problem: Calculate the electric field (everywhere in space) due to a uniformly-charged sphere
- Solid sphere of radius $R$
- Volume charge density: $\rho = \frac{Q}{V} = \frac{Q}{\frac{4\pi}{3} R^3}$

Solution by Coulomb’s Law
- We know the $E$ due to a point charge $dq$ by Coulomb’s Law
- Integrate over the volume of the sphere
- Repeat for inside ($r < R$) and outside ($r > R$) of the sphere

Of course you can do it, but not fun
Apply Gauss’s Law

Consider an imaginary sphere $S$ of radius $r$
- Concentric with the charged sphere

Symmetry of the problem tells us that $E$ field is radial in direction, and its magnitude is constant over $S$
- Think about it …

Apply Gauss’s Law to $S$
\[
\Phi = \int_S E \cdot da = 4\pi Q
\]
- The integral is easy:
\[
\int_S E(r) da = E(r) \cdot 4\pi r^2
\]

\[
E(r) = \frac{Q}{r^2}
\]

This is for $r > R$

Apply Gauss’s Law

Now make $S$ smaller than the charged sphere
The symmetry argument still holds
Apply Gauss’s Law
\[
\Phi = \int_S E \cdot da = 4\pi q_{\text{inside}}
\]
- Charge inside $S$ is $q_{\text{inside}} = \frac{r^3}{R^3} Q$
- Integral is the same as before:
\[
\int_S E(r) da = E(r) \cdot 4\pi r^2
\]

\[
E(r) = \frac{Qr}{R^3}
\]

This is for $r < R$
Am I Done?

One point remains before we can get full credit
- We were asked to determine the electric field ➔ a vector
- We need to specify the direction!

Complete solution:

\[
E = \begin{cases} 
\frac{Q}{r^2} \hat{r} & \text{for } r \geq R \\
\frac{Qr}{R^2} \hat{r} & \text{for } r < R 
\end{cases}
\]

- E field outside a charged sphere = E field for a point charge
- NB: the same holds for gravity

Checklist for E&M Problems

Read the problem
Look for symmetries:
- Which coordinate system works best?
- What cancels out?
- Which way the vectors should point?
Look for ways to avoid integration
Turn the math crank
Write down the complete solution
- e.g. magnitudes and directions for all the different regions
Read the problem again – Did you answer what is asked?
Box the solution: your TF will be grateful
Infinite Sheet of Charge

Problem: Calculate the electric field at a distance $z$ from a positively charged infinite plane

- Surface charge density: $\sigma = \frac{\text{charge}}{\text{area}}$

Use Gauss again

- Which surface to use?
  - What symmetry do we have?
  - Consider a cylinder $\Rightarrow$
    - Area $A$ and height $2z$

$E$ field must be vertical

- How do we know that?

Infinite Sheet of Charge

Total flux $\Phi_{\text{total}} = \Phi_{\text{top}} + \Phi_{\text{side}} + \Phi_{\text{bottom}}$

- Side is parallel to $E \Rightarrow E \cdot da = 0 \Rightarrow$ No flux
- Top and bottom are symmetric $\Rightarrow$ Same flux

$\Phi_{\text{total}} = 2\Phi_{\text{top}} = 2AE(z)$

Charge inside the cylinder is

$q_{\text{inside}} = A\sigma$

Using Gauss $\Rightarrow E(z) = 2\pi\sigma$

- Don’t forget the direction!

$E = \begin{cases} 
+2\pi\sigma\hat{z} & \text{for } z > 0 \\
-2\pi\sigma\hat{z} & \text{for } z < 0 
\end{cases}$

The result is worth remembering:

Infinite sheet of charge produces uniform $E$ field of $2\pi\sigma$ above and below
Pair of Charged Sheets

Place two oppositely-charged large sheets in parallel
- Consider an area $A$ of them
- $E$ fields from the two sheets overlap and add up
  - Between the sheets: $E = 4\pi\sigma$
  - Cancel each other outside
- Two sheets also attract each other (obviously)
  - Top sheet feels
  
  $$E_{\text{bottom}} = -2\pi\sigma \hat{z}$$

  - The force on area $A$ of the top sheet is
  
  $$F = \sigma A E_{\text{bottom}} = -2\pi\sigma^2 A \hat{z} = -\frac{E^2}{8\pi} A \hat{z}$$

Imagine we move the top sheet upward by a distance $d$
- We must do work $W = Fd = \frac{E^2}{8\pi} Ad$
- The energy of the system increases by $W$

Q: Where exactly is this energy?
- Note that the volume of the space between the sheets increased by $Ad$
- This is also where $E$ field exists

Space with $E$ holds energy with a volume density $u = \frac{E^2}{8\pi}$

Total electrostatic energy of a system is

$$U = \int \frac{E^2}{8\pi} dV$$
### Summary

**Electric field**
- Defined by $\mathbf{F} = q\mathbf{E}$
- Can be expressed by field lines

**Flux of electric field**
- Defined by $\Phi = \int \mathbf{E} \cdot d\mathbf{a}$
- Note the sign convention: positive if coming out

**Gauss's Law**
- $\int \mathbf{E} \cdot d\mathbf{a} = 4\pi q$
- Useful for solving $\mathbf{E}$ fields with symmetries
  - Spherical charge distribution and infinite sheet
  - Infinite sheet generates uniform $\mathbf{E}$ field of $2\pi \sigma$

**Energy density** of electric field
- $u = \frac{E^2}{8\pi}$
- $U = \int \frac{E^2}{8\pi} \, dV$