What We Did Last Time

A dipole generates electric field \( E_r = \frac{2p \cos \theta}{r^3}, E_\theta = \frac{p \sin \theta}{r^3} \)

A dipole in an electric field receives:
- Torque \( \mathbf{N} = \mathbf{p} \times \mathbf{E} \)
- If \( \mathbf{E} \) is non-uniform, net force \( \mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E} \)
  - Dipoles are attracted to stronger \( \mathbf{E} \) field

Density of polarization \( \mathbf{P} = N \mathbf{p} \)
- Small volume \( dv \) of dielectric looks like a dipole \( \mathbf{P} dv \)
- A cylinder parallel to \( \mathbf{P} \) looks like charge density \( \pm \mathbf{P} \) on the ends
- Average electric field inside the cylinder is \( \langle \mathbf{E} \rangle = -4\pi \mathbf{P} \)

A uniformly polarized sphere
- External field looks like produced by a dipole \( \mathbf{VP} \)
- Internal field \( \mathbf{E} = -(4\pi/3) \mathbf{P} \)
Today’s Goals

Connect polarization $P$ and the dielectric constant $\varepsilon$
Continue discussion of polarized sphere
  ○ Boundary condition at the surface
  ○ How to make a sphere polarized uniformly
Examine the effects of non-uniform polarization
  ○ Non-uniform $P$ creates “bound” charge distribution
  ○ Charge screening, electric displacement
Discuss time-dependent $E$ field in dielectrics
  ○ Modified Maxwell’s equations
  ○ Electromagnetic waves in dielectrics

Dielectric Constant

How does $P$ relate to the dielectric constant $\varepsilon$?
Consider the filled capacitor again
  ○ Electric field is reduced by factor $1/\varepsilon$
    $$E = \frac{4\pi\sigma}{\varepsilon} = 4\pi\sigma - 4\pi P$$

Field from charge on the plates
Field from polarization

$$E = \varepsilon E - 4\pi P$$

$$\frac{P}{E} = \frac{\varepsilon - 1}{4\pi} \equiv \chi_e$$

Electric susceptibility

Polarization density $P$ is related to the average electric field $E$ that causes it by:
  ○ This is an empirical law, and is “correct” within limits
    $$P = \chi_e E = \frac{\varepsilon - 1}{4\pi} E$$
Uniformly Polarized Sphere

A dielectric sphere is uniformly polarized along +z
- It contains dipoles \( p = q s \) with density \( N \)
  \[ \mathbf{P} = N \mathbf{p} = N q s \]
- This can be seen as two overlapping spheres
  - Charge densities are \(+Nq\) and \(−Nq\)
  - Centers are separated by distance \(s\)
- From outside, each sphere looks like a point charge (recall Gauss) \(+NqV\) and \(−NqV\)

Field outside is identical to that generated by a single dipole moment \(NqVs = \mathbf{V.P}\)
- We know this field from the last lecture:
  \[ \varphi(r > R) = \frac{\mathbf{V.P} \cdot \hat{r}}{r^2} = \frac{\mathbf{V.P} \cos \theta}{r^2} \]

Uniformly Polarized Sphere

Inside the sphere, there is no net charge
→ The field must obey Laplace’s eqn.
- We also need the boundary condition, i.e., values of \(\varphi\) at the surface, which we know from the outside field
  \[ \varphi(r = R) = \frac{\mathbf{V.P} \cos \theta}{R^2} = \frac{\mathbf{V}}{R^3} \mathbf{P} \cos \theta = \frac{4\pi}{3} \mathbf{Pz} \]
- A uniform electric field along +z works
  \[ \varphi(r < R) = \frac{4\pi}{3} \mathbf{Pz} \quad \Rightarrow \quad \mathbf{E}(r < R) = -\frac{4\pi}{3} \mathbf{P} \]

The field due to a uniformly polarized sphere is
- Inside: \(\mathbf{E} = -(4\pi/3)\mathbf{P}\)
- Outside: identical to the field generated by a dipole \((4\pi/3)R^3\mathbf{P}\)
**Boundary Conditions**

Electric potential $\phi$ is a continuous function of space
- Otherwise there would be infinite electric field

As a result, electric field has to satisfy the following conditions on the surface of the dielectric
- $E_\parallel$ parallel to the surface is continuous
- $E_\perp$ perpendicular to the surface may be discontinuous

Check this with the uniformly polarized sphere:

For outside:
$$\begin{cases} E_r = \frac{8\pi}{3} P \cos \theta \\ E_\theta = \frac{4\pi}{3} P \sin \theta \end{cases}$$

For inside:
$$\begin{cases} E_r = -\frac{4\pi}{3} P \cos \theta \\ E_\theta = \frac{4\pi}{3} P \sin \theta \end{cases}$$

**Dielectric Sphere in E Field**

How does a dielectric sphere get uniformly polarized?
- Try the simplest way — put it in a uniform external field $E_0$
- Suppose this leads to a uniform polarization $P$
- $P$ generates a uniform field inside: $E' = -\frac{4\pi}{3} P$

Total field inside is $E_{\text{inside}} = E_0 + E' = E_0 - \frac{4\pi}{3} P$

Resulting polarization is
$$P = \chi_0 E_{\text{inside}} = \frac{\varepsilon - 1}{4\pi} \left( E_0 - \frac{4\pi}{3} P \right)$$

We can solve this to find
$$E' = \left( \frac{3}{\varepsilon + 2} \right) E_0$$

and
$$P = \frac{3}{4\pi} \left( \frac{\varepsilon - 1}{\varepsilon + 2} \right) E_0$$

- Uniform external field polarizes the sphere uniformly
Bound Charge

Consider a small area $da$ inside a dielectric
- Polarization $P = Nq\sigma$
  - How many dipoles “straddle” this area?

- Charge $qN\sigma \cdot da = P \cdot da$ is split from the corresponding negative charge by the area $da$

Integrate this over a closed surface $S$
- How much (negative) charge remains inside?
  $$ Q = -\int_S P \cdot da $$

- Use the Divergence Theorem
  $$ \rho = -\text{div} P $$
  “Bound” charge distribution due to non-uniform polarization

Free and Bound Charges

Inside dielectric, two “types” of charges may exist:
- “Bound” charge $\rho_{\text{bound}}$ belongs to the dielectric material
  - Appears only when $E$ field polarizes the dielectric
    $$ \rho_{\text{bound}} = -\text{div} P \quad P = \frac{\epsilon - 1}{4\pi} E $$

- “Free” charge $\rho_{\text{free}}$ is brought in from outside

Both “bound” and “free” charges create $E$ field
$$ \text{div} E = 4\pi (\rho_{\text{free}} + \rho_{\text{bound}}) $$

- For a given $\rho_{\text{free}}$ distribution and a constant $\epsilon$
  $$ \text{div} E = 4\pi \rho_{\text{free}} - (\epsilon - 1) \text{div} E $$
  $$ \text{div} E = \frac{4\pi \rho_{\text{free}}}{\epsilon} $$

$E$ follows Gauss’s Law with $\rho_{\text{free}}$ except for a factor $1/\epsilon$
Screening

A point charge $Q$ is inside a dielectric
- $Q$ is the only "free" charge here

Electric field at distance $r$ from the charge is
$$ E = \frac{Q}{\varepsilon r^2} \hat{r} $$
Coulomb reduced by $1/\varepsilon$
- Polarization density $P$ due to this field is
$$ P = \frac{\varepsilon - 1}{4\pi} E = \frac{(\varepsilon - 1)Q}{4\pi \varepsilon r^2} \hat{r} $$
- This creates bound charge density $\rho_{\text{bound}} = \nabla \cdot P = \frac{(\varepsilon - 1)Q}{4\pi \varepsilon} \left( \frac{\hat{r}}{r^2} \right)$
- This div is zero everywhere except at the origin
- Integrate inside a very small sphere around $Q$
$$ \int_V \rho_{\text{bound}} dV = -\int_S P \cdot d\mathbf{a} = -\frac{\varepsilon - 1}{\varepsilon} Q $$
Negative charge surrounds $Q$

Polarization creates a "screen" around free charges

Electric Displacement

Electric displacement $D$ is defined by $D \equiv E + 4\pi P$
- For electric field inside an isotropic dielectric
$$ P = \frac{\varepsilon - 1}{4\pi} E \rightarrow D = \varepsilon E $$
- $D$ satisfies Gauss's Law with free charge: $\nabla \cdot D = 4\pi \rho_{\text{free}}$

Importance of $D$ is more historic than practical
- It’s easy to calculate only in linear, isotropic dielectric
  - In that case, writing $D$ instead of $\varepsilon E$ saves only a little ink
- In a more complex medium, it’s safer to use $E$ and $4\pi P$, and keep track of how the material reacts to $E$
Bound-Charge Current

In a non-static system, $\mathbf{P}$ may vary with time

- Imagine $\mathbf{s}$ changing in response to changing $\mathbf{E}$
- Charges $+q$ and $-q$ move $\rightarrow$ “bound” current

$$
\mathbf{J}_{\text{bound}} = N \left( q \frac{dr}{dt} - q \frac{dr}{dt} \right) = Nq \frac{ds}{dt} = \frac{\partial \mathbf{P}}{\partial t}
$$

$\mathbf{J}_{\text{bound}}$ adds to the “free” current $\mathbf{J}$ in generating $\mathbf{B}$

$$
\nabla \times \mathbf{B} = \frac{4\pi}{c} \left( \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} \right) + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial }{\partial t} \left( \mathbf{E} + 4\pi \mathbf{P} \right)
$$

For a linear isotropic dielectric,

$$
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t}
$$

Electromagnetic Waves

Inside a dielectric with no free charge and no free current

$$
\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}
$$

$$
\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t}
$$

- Same technique used in Lecture #18 turn them into
  $$
  \nabla^2 \mathbf{E} = \frac{\varepsilon}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \frac{\varepsilon}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}
  $$

- Solutions are waves propagating with speed $c/\sqrt{\varepsilon} < c$

**EM waves travel slower in a dielectric by factor** $n = \sqrt{\varepsilon}$

- $n$ is the **index of refraction** of the material
Electromagnetic Waves

Plane wave solutions of the Maxwell’s eqns. are
\[ E = E_0 \sin(k \cdot r - \omega t) \quad B = B_0 \sin(k \cdot r - \omega t) \]

- Wave equation:
  \[ \nabla^2 E = \frac{\varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} \quad k^2 = \frac{\varepsilon}{c^2} \omega^2 \quad \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon}} \]

- Divergence:
  \[ \nabla \cdot E = 0 \quad \nabla \cdot B = 0 \quad \mathbf{k} \perp \mathbf{E}_0 \quad \mathbf{k} \perp \mathbf{B}_0 \]

- Curl:
  \[ \nabla \times E = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \mathbf{k} \times \mathbf{E}_0 = \frac{\omega}{c} \mathbf{B}_0 \quad \hat{\mathbf{k}} \times \mathbf{E}_0 = \frac{1}{\sqrt{\varepsilon}} \mathbf{B}_0 \]

Similar to the vacuum solutions except:
- Propagation velocity is reduced by \( \frac{1}{\sqrt{\varepsilon}} \)
- \(|E|\) is smaller than \(|B|\) by the same factor

Frequency Dependence

Discussion so far applies to any dielectric = any insulator
- All insulators are transparent, with \( n = \sqrt{\varepsilon} \)
- Index of refraction of water is \( \sqrt{80} = 8.9 \) \[ \text{Wrong!} \]

When \( E \) changes, it takes time for dielectrics to polarize
- Dielectric “constant” is constant only for static/slowly-changing field

Especially true for liquid of polar molecules, e.g. water
- Molecules must rotate \( \Rightarrow \) Takes \( \sim 10^{-11} \) seconds
- \( \varepsilon \) large up to \( 10^{10} \) Hz, then drops to an “ordinary” value of 1.78
- Index of refraction of water for visible light is 1.33
Ordinary Dielectrics

Inside insulators are electrons bound to atoms

- They behave as mass-spring oscillators
- EM waves drive them with \( F = -eE \)
- If the frequency \( \omega \) is close to the resonance frequency \( \omega_0 \), the electrons oscillate strongly
- They absorb the incoming waves

Typical \( \omega_0 \) for bound electrons are \( 10^{15} - 10^{16} \) Hz

- In the visible to ultraviolet (UV) region
- Loosely bound electrons (small \( k \rightarrow \) small \( \omega_0 \)) make material opaque in visible light

Positively charged hydrogen atoms in molecules oscillate at lower \( \omega_0 \) because of larger \( m \)

- Microwave is absorbed by water and organic compounds

Summary

Electric susceptibility of a dielectric \( \mathbf{P} = \chi_e \mathbf{E} = \frac{\varepsilon - 1}{4\pi} \mathbf{E} \)

- At boundaries of dielectrics, \( \mathbf{E} \parallel \) is continuous, but \( \mathbf{E} \perp \) may not be

If \( \mathbf{P} \) is not uniform, bound charge \( \rho_{\text{bound}} = -\text{div} \mathbf{P} \) appears

\[
\text{div} \mathbf{E} = 4\pi (\rho_{\text{free}} + \rho_{\text{bound}}) = \frac{4\pi \rho_{\text{free}}}{\varepsilon} \quad \text{for linear, isotropic dielectric}
\]

In such a medium, Maxwell's equation is modified as:

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t}
\]

- Wave solutions propagate with a reduced speed \( c/\sqrt{\varepsilon} \)
- Frequency dependence of \( \varepsilon \) makes the solutions more complex