Electromagnetism
Physics 15b

Lecture #5
Curl
Conductors

Purcell 2.13–3.3

What We Did Last Time

Defined divergence: \( \text{div} \mathbf{F} = \lim_{V \to 0} \frac{\int_{S} \mathbf{F} \cdot d\mathbf{a}}{V} = \mathbf{\nabla} \cdot \mathbf{F} = \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \)

Gauss’s Law (local version): \( \nabla \cdot \mathbf{E} = 4\pi \rho \)

- Linked to the integral version by the Divergence Theorem: \( \int_{S} \mathbf{F} \cdot d\mathbf{a} = \int_{V} \nabla \cdot \mathbf{F} dV \)

Defined the Laplacian: \( \nabla^2 f = \nabla \cdot (\nabla f) = \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \)

- From Gauss’s Law: \( 4\pi \rho = -\nabla^2 \phi \)

Laplace’s equation: \( \nabla^2 \phi = 0 \)

- Average theorem, no-max/min theorem, impossibility theorem, uniqueness theorem
Today’s Goals

Define curl of vector field
- Start from line integral around a loop and shrink
- Stokes’ Theorem
- A few examples to familiarize ourselves

Discuss conductors
- Finally a real "stuff" ???

Consider good conductors holding electric charges in a static condition
- How do the charges distribute?
- What electric field, potential should appear?
- What do you mean by “good”?

3-D, 2-D, 1-D, 0-D

We have

\[ \int_V \nabla \cdot \mathbf{F} \, dV = \int_S \mathbf{F} \cdot d\mathbf{a} \]

\[ \int_{P_1}^{P_2} \nabla f \cdot d\mathbf{s} = f(P_2) - f(P_1) \]

- 3-D integral linked to 2-D integral over the boundary
- 1-D integral linked to values at end points (= 0-D)

Shouldn’t there be a similar relation between 2-D and 1-D?
- It could be a surface integral and a line integral around its border
Circulation

Consider a closed (loop) path $C$ in space where a vector field $\mathbf{F}$ exists.

- Define circulation as the line integral
  $$\Gamma = \int_C \mathbf{F} \cdot d\mathbf{s}$$
  - This is zero for static electric field

Split $C$ into two sub-loops $C_1$ and $C_2$ with a “bridge”

- The integrals along the bridge cancel
  $$\Gamma = \Gamma_1 + \Gamma_2 = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

We can continue splitting into ever smaller loops:
  $$\Gamma = \sum_j \Gamma_j$$

Curl

As we make the area $a_j$ of each loop smaller, the line integral $\Gamma_j$ shrinks as well

- The ratio $\Gamma_j/a_j$ remains finite
  $$\lim_{a_j \to 0} \frac{\int_{C_j} \mathbf{F} \cdot d\mathbf{s}}{a_j}$$

For a very small loop, the shape does not matter, but the orientation does

- Define the orientation vector $\mathbf{n}$ of the loop by the right-hand rule
- At a given point, draw three loops with $\mathbf{n} = \hat{x}, \hat{y}, \hat{z}$
- Calculate the limits and build a vector $\Rightarrow \text{curl} \text{ of } \mathbf{F}$

Curl $\mathbf{F}$ is a vector field, representing the circulation per unit area in three possible orientations.
Stokes’ Theorem

Going back to splitting circulation, in the limit of small loops

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \lim_{a_i \to 0} \sum_j \int_{C_j} \mathbf{F} \cdot d\mathbf{s} = \lim_{a_i \to 0} \sum_j \text{curl} \mathbf{F} \cdot a_j = \int_S \text{curl} \mathbf{F} \cdot d\mathbf{a} \]

- This is Stokes’ Theorem
- Like Gauss’s Divergence Theorem, this is math, not physics

Applying our (physics) knowledge that line-integral of electric field is path independent, we get

\[ \int_C \mathbf{E} \cdot d\mathbf{s} = 0 \text{ for any closed path } C \quad \Rightarrow \quad \text{curl} \mathbf{E} = 0 \]

- So far so good

Q: What exactly is a curl and how do I calculate it?

Curl in Water

You are swimming in a river, with velocity of water flow \( \mathbf{v} \)

If you swim in a loop and come back to the starting point, are you helped, or impeded, by the flow?
- That’s circulation
  \[ \int_C \mathbf{v} \cdot d\mathbf{s} > 0 \rightarrow \text{helped}, \quad < 0 \rightarrow \text{impeded} \]

If you drop a small leaf in the water, will it rotate?
- That’s curl
  \[ \text{curl} \mathbf{F} \neq 0 \rightarrow \text{rotates} \]

If you drop a thousand leaves everywhere, you can tell which way you should swim by Stokes’ Theorem
Focus on the z-component of curl $\mathbf{F}$
- Draw a rectangular loop on the x-y plane
- For small $\Delta x$, $\Delta y$, we can approximate the line-integral on each side by (length) $\times$ (value of $\mathbf{F}$ in the middle)
  = $\Delta x \mathbf{F}_x(x+\Delta x,y,z) + \Delta y \mathbf{F}_y(x,y+\Delta y,z)$
  = $\Delta x \mathbf{F}_x(x,y+\Delta y,z) - \Delta y \mathbf{F}_y(x+\Delta x,y,z)$
  = $\Delta x \Delta y \left( \frac{\partial \mathbf{F}_x}{\partial y} + \frac{\partial \mathbf{F}_y}{\partial x} \right)$
- Dividing by the area $\Delta x \Delta y$  \begin{align*} \text{curl} \mathbf{F} \cdot \hat{z} &= \frac{\partial \mathbf{F}_y}{\partial x} - \frac{\partial \mathbf{F}_x}{\partial y} \end{align*}

x- and y-components are similar

Some people find it easier to remember this as a determinant: \begin{align*} \text{curl} \mathbf{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \nabla \times \mathbf{F} \end{align*}

Cylindrical: \begin{align*} \text{curl} \mathbf{F} &= \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} & \frac{1}{r} \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} & \frac{1}{r} \frac{\partial F_\theta}{\partial r} - \frac{\partial F_r}{\partial \theta} \end{vmatrix} \hat{z} \end{align*}

Spherical: \begin{align*} \text{curl} \mathbf{F} &= -\frac{1}{r \sin \theta} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial F_\phi}{\partial r} & \frac{1}{r} \frac{\partial F_\theta}{\partial \phi} - \frac{\partial F_r}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial F_\theta}{\partial \phi} - \frac{\partial F_\phi}{\partial \theta} \end{vmatrix} \hat{\phi} \end{align*}
**Examples**

\[ F_y = ax \]
\[ F_x = -ay, F_y = ax \]
\[ (\nabla \times F)_z = \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial y} = a \]
\[ (\nabla \times F)_z = 2a \]

\[ F_y = -\frac{ay}{\sqrt{x^2 + y^2}} \]
\[ F_x = \frac{ax}{\sqrt{x^2 + y^2}} \]
\[ \nabla \times F = 0 \]

**Curl of Gradient**

The curl of the gradient of a scalar field is \( \nabla \times (\nabla f) = 0 \)

- Check explicitly: \( (\nabla \times (\nabla f)) \cdot \hat{x} = \frac{\partial (\nabla f_x)}{\partial x} - \frac{\partial (\nabla f_y)}{\partial y} = 0 \)
- Use Stokes’ Theorem: \( \int_S (\nabla \times (\nabla f)) \cdot \mathbf{a} = \int_C \nabla f \cdot d\mathbf{s} = 0 \)

If a vector field (like \( \mathbf{E} \)) can be expressed as a gradient of a scalar field (like \( \phi \)), it must be curl-free.

On the other hand, any curl-free vector field can be expressed as a gradient of a scalar field.

- If \( \nabla \times \mathbf{F} = 0 \), by Stokes it’s line-integral must be path-independent.
- Line integral from an arbitrary chosen reference point will do

\[ f(\mathbf{r}) = \int_0^n \mathbf{F} \cdot d\mathbf{s} \quad \rightarrow \quad \mathbf{F} = \nabla f(\mathbf{r}) \]
3-D, 2-D, 1-D, 0-D

\[
\int \nabla \cdot \mathbf{F} \, dV = \int \mathbf{F} \cdot d\mathbf{a}
\]

\[
\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\]

\[
\nabla \times \mathbf{F} = \hat{x} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
\]

\[
\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\]

\[
\nabla \times \mathbf{F} = \hat{x} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
\]

Conductor

Conductor vs. insulator is a matter of speed
- When electric field is applied to any material, charged particles inside move
- Good conductor: many charges, quick to move
- Good insulator: few charges, slow to move

Conductivity = a measure of goodness of conductor

\[
\sigma = \frac{n \cdot v}{E} = \frac{(\text{charge density}) \times (\text{average velocity})}{\text{(applied electric field)}}
\]

- Unit in CGS: \(\text{esu/cm}^3 \cdot (\text{cm/s}) = \frac{1}{\text{s}}\), in SI: \(\text{C/m}^3 \cdot (\text{m/s}) = \frac{1}{\Omega \cdot \text{m}}\)

- Will define this properly in Lecture 8
Conductivity

Conductivity $\sigma$ of material span 23 orders of magnitude

They fall into three broad groups:

- $\sigma >> 1 \Rightarrow$ conductors
  - Mostly metals, in which free electrons act as the charge carriers
  - $\sigma = O(1) \Rightarrow$ semiconductors
  - $\sigma << 1 \Rightarrow$ insulators

In electrostatics, where time is not an issue, we only have to consider two extreme cases:

- Perfect conductor: $\sigma = \infty$
- Perfect insulator: $\sigma = 0$

<table>
<thead>
<tr>
<th>Material</th>
<th>$\sigma$ [1/Ωm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>silver</td>
<td>$6.3 \times 10^7$</td>
</tr>
<tr>
<td>copper</td>
<td>$5.9 \times 10^7$</td>
</tr>
<tr>
<td>manganin alloy</td>
<td>$2.3 \times 10^6$</td>
</tr>
<tr>
<td>silicon</td>
<td>33</td>
</tr>
<tr>
<td>salt water</td>
<td>23</td>
</tr>
<tr>
<td>drinking water</td>
<td>0.0005 – 0.05</td>
</tr>
<tr>
<td>deionized water</td>
<td>$5.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>wood</td>
<td>$10^{-8} – 10^{-11}$</td>
</tr>
<tr>
<td>glass</td>
<td>$10^{-10} – 10^{-14}$</td>
</tr>
<tr>
<td>sulfur</td>
<td>$5 \times 10^{-16}$</td>
</tr>
<tr>
<td>rubber</td>
<td>$10^{-13} – 10^{-16}$</td>
</tr>
</tbody>
</table>

Source: J.E. Hoffman

Field Inside Conductor

As long as we only consider a static condition (“electrostatic equilibrium”) in which no charge is moving, there is no electric field inside conductors

- If $E \neq 0$, then the charges would be still moving

When a conductor is placed in an external field $E_{ext}$,

- charges move inside because of $F = qE$
- charge distribution $\rho$ becomes non-zero
- internal electric field $E_{int}$ is generated

and the process continues until $E_{ext} + E_{int} = 0$

Sounds a little involved?

- This is not really different from any system of fluids settling into the lowest-energy state
### Potential and Charge Density

Since \( \mathbf{E} = 0 \) inside a conductor, \( \Delta \phi_{BA} = \int_{A}^{B} \mathbf{E} \cdot d\mathbf{s} = 0 \) for any two points connected by a conductor.

- The entire volume (and surface) of a contiguous piece of conductor is an equipotential.

Also, there is no net charge density inside a conductor because \( 4\pi \rho = \nabla \cdot \mathbf{E} = 0 \):

- All electric charges, if any, must be on the surface.

**NB:** There are electrons and protons inside the conductor. Their densities are balanced so that there is no net charge after averaging over small volumes.

### Electric Field Near Surface

Consider a conductor with surface charge density \( \sigma \):

- What is the \( \mathbf{E} \) field immediately outside?

We know the surface is an equipotential:

- \( \mathbf{E} \) field must be perpendicular to it.

Draw a cylinder half buried into the surface:

- Charge inside the cylinder is \( \sigma A \)
- \( \mathbf{E} = 0 \) at the bottom (inside)
- \( \mathbf{E} \) is parallel to the side of the cylinder
- Gauss’s Law: \( 4\pi \sigma A = \int_{\text{top}} \mathbf{E} \cdot d\mathbf{a} = EA \)

\[ \mathbf{E} = 4\pi \sigma \hat{n} \quad \text{where } \hat{n} \text{ is } \perp \text{ to the surface} \]
Summary

Defined curl of vector field by \( \text{curl} \mathbf{F} \cdot \mathbf{n} \equiv \lim_{a \to 0} \frac{\int_C \mathbf{F} \cdot d\mathbf{s}}{a} \)

- In Cartesian: \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \hat{x} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \)

Stokes' Theorem: \( \int_{\text{surface}} \nabla \times \mathbf{F} \, dV = \int_{\text{loop}} \mathbf{F} \cdot d\mathbf{s} \)

Studied (ideal) conductor in electrostatic equilibrium
- \( \mathbf{E} = 0 \) inside, \( \phi = \text{const} \), no net charge density inside
- Field immediately outside is perpendicular to the surface and \( E = 4\pi \sigma \), where \( \sigma \) is the surface charge density