Lecture 4
Coupled Oscillators
(→ H&L Section 4.1)
What We Did Last Time

- Studied forced oscillation
  - Solved the equation of motion with friction and found the steady-state solution
  - Oscillation becomes large near the resonance frequency
  - Phase changes from $0 \rightarrow -\pi/2 \rightarrow -\pi$ as the frequency increases through the resonance
  - Work done by the force is consumed by the friction
  - Energy consumption is large near the resonance
Goals for Today

- Coupled Oscillators
  - How a pair of harmonic oscillators behave when they are connected with each other.
- Will use some linear algebra
- Prepare for real waves
Coupled Oscillators

- Two identical pendulums are connected by a spring
  - Consider small oscillation
    $$x_1 \approx L \theta_1 \quad x_2 \approx L \theta_2$$
  - Spring tension is
    $$F_s = -k(x_1 - x_2)$$
  - Restoring force from gravity
    $$F_1 = mg \sin \theta_1 = \frac{mg}{L} x_1 \quad F_2 = \frac{mg}{L} x_2$$

Equation of motion
Equation of Motion

\[
\begin{align*}
\frac{d^2 x_1}{dt^2} &= -\frac{mg}{L} x_1 - k(x_1 - x_2) \\
\frac{d^2 x_2}{dt^2} &= -\frac{mg}{L} x_2 - k(x_2 - x_1)
\end{align*}
\]

- Must solve two differential equations simultaneously
  - Brute force
  - Symmetry
  - Linear algebra
Brute Force

\begin{align*}
\left\{ \begin{array}{l}
m \frac{d^2 x_1}{dt^2} &= - \frac{mg}{L} x_1 - k(x_1 - x_2) \\
m \frac{d^2 x_2}{dt^2} &= - \frac{mg}{L} x_2 - k(x_2 - x_1)
\end{array} \right.

x_2 &= \frac{1}{k} \left[ m \frac{d^2 x_1}{dt^2} + \left( \frac{mg}{L} + k \right) x_1 \right]

\frac{m^2}{k} \frac{d^4 x_1}{dt^4} + 2 \left( \frac{m^2 g}{kL} + m \right) \frac{d^2 x_1}{dt^2} + \left( \frac{1}{k} \left( \frac{mg}{L} \right)^2 + 2 \frac{mg}{L} \right) x_1 = 0

\text{Let } x_1 = e^{xt}

\left[ \frac{m^2}{k} X^4 + 2 \left( \frac{m^2 g}{kL} + m \right) X^2 + \frac{mg}{L} \left( \frac{mg}{kL} + 2 \right) \right] e^{xt} = 0
\end{align*}
Brute Force

\[ X^4 + 2\left( \frac{g}{L} + \frac{k}{m} \right)X^2 + \frac{g}{L}\left( \frac{g}{L} + 2\frac{k}{m} \right) = 0 \]

\[ \left( X^2 + \frac{g}{L} \right)\left( X^2 + \frac{g}{L} + 2\frac{k}{m} \right) = 0 \]

For each solution of \( X \), we get

\[ x_1 = e^{xt} \]

Then we put it into

\[ x_2 = \frac{1}{k} \left[ m \frac{d^2x_1}{dt^2} + (\frac{mg}{L} + k)x_1 \right] \]

\[ X = \pm i\sqrt{\frac{g}{L}}, \pm i\sqrt{\frac{g}{L} + 2\frac{k}{m}} \]

What a pain…
The two equations are symmetric

By adding & subtracting them, you get

\[
\begin{align*}
& m \frac{d^2 x_1}{dt^2} = -\frac{mg}{L} x_1 - k(x_1 - x_2) \\
& m \frac{d^2 x_2}{dt^2} = -\frac{mg}{L} x_2 - k(x_2 - x_1)
\end{align*}
\]

We can solve the equations for \((x_1 + x_2)\) and \((x_1 - x_2)\)

Looks like simple harmonic oscillators
Symmetry

Solutions are:

\[
\begin{cases}
\frac{d^2(x_1 + x_2)}{dt^2} = -\frac{g}{L}(x_1 + x_2) \\
\frac{d^2(x_1 - x_2)}{dt^2} = -\left(\frac{g}{L} + 2\frac{k}{m}\right)(x_1 - x_2)
\end{cases}
\]

\[
\begin{align*}
\omega_p &= \sqrt{\frac{g}{L}} \\
\omega_s &= \sqrt{\frac{g}{L} + 2\frac{k}{m}}
\end{align*}
\]

- How do they look like?
"Parallel" Oscillation

\[
\begin{align*}
    x_1 + x_2 &= Ae^{i\omega_P t} \\
    x_1 - x_2 &= Be^{i\omega_S t}
\end{align*}
\]

\[\omega_P = \sqrt{\frac{g}{L}}\]

- Let \( B = 0 \). \( \Rightarrow \) \( x_1 - x_2 = 0 \)
  - Two pendulums are moving in parallel
  - The spring does nothing
  - \( \omega_P \) = natural frequency of free pendulum without coupling
**“Symmetric” Oscillation**

\[
\begin{cases}
x_1 + x_2 = Ae^{i\omega pt} \\
x_1 - x_2 = Be^{i\omega_st}
\end{cases}
\]

\[
\omega_s = \sqrt{\frac{g}{L} + 2\frac{k}{m}}
\]

- Let \( A = 0 \) \( \Rightarrow \) \( x_1 + x_2 = 0 \)
  - Two pendulums are moving symmetrically
  - The spring gets expanded/shrunk by twice the movement of each pendulum
  - \( \omega_s \) is determined by both the pendulums and the spring
Normal Modes

- The two oscillating patterns are called the **normal modes**
  - Both are simple harmonic oscillation
  - Constant frequency & amplitude
- **Two normal modes** for **two coupled** oscillators
  - Two pendulums have **two initial conditions** each \((x_i \text{ and } dx_i/dt)\)
  - Two normal modes have **two parameters** each \((a \cos \omega t + b \sin \omega t)\)
Once you know the normal modes, the general solution is a linear combination of them.

\[
\begin{align*}
    x_1 + x_2 &= Ae^{i\omega_p t} \\
    x_1 - x_2 &= Be^{i\omega_s t}
\end{align*}
\]

Let’s find a solution that satisfies an initial condition:

\[
\begin{align*}
    x_1(0) &= a, \quad \dot{x}_1(0) = 0 \\
    x_2(0) &= 0, \quad \dot{x}_2(0) = 0
\end{align*}
\]

\[ A = B = a \]
Specific Solution

- Take the real part:
  \[
  \begin{align*}
  x_1 &= \frac{a}{2} (\cos \omega_p t + \cos \omega_s t) \\
  x_2 &= \frac{a}{2} (\cos \omega_p t - \cos \omega_s t)
  \end{align*}
  \]

- Use
  \[
  \begin{align*}
  \cos \alpha + \cos \beta &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\
  \cos \alpha - \cos \beta &= -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}
  \end{align*}
  \]

- Hence,
  \[
  \begin{align*}
  x_1 &= a \cos \left( \frac{\omega_p + \omega_s}{2} t \right) \cos \left( \frac{\omega_p - \omega_s}{2} t \right) \\
  x_2 &= -a \sin \left( \frac{\omega_p + \omega_s}{2} t \right) \sin \left( \frac{\omega_p - \omega_s}{2} t \right)
  \end{align*}
  \]
Specific Solution

\[ \omega_s = 1.1 \times \omega_p \]
Specific Solution

\[
\begin{align*}
    x_1 &= \frac{a}{2} \cos \left( \frac{\omega_P + \omega_S}{2} t \right) \cos \left( \frac{\omega_P - \omega_S}{2} t \right) \\
    x_2 &= \frac{a}{2} \sin \left( \frac{\omega_P + \omega_S}{2} t \right) \sin \left( \frac{\omega_P - \omega_S}{2} t \right)
\end{align*}
\]

- Plot shows the case where \( \omega_P \) and \( \omega_S \) are very close

\[
|\omega_P - \omega_S| \ll \omega_P + \omega_S
\]

- The spring constant is small = weak coupling between the two pendulums

\[
\begin{align*}
    \omega_P &= \sqrt{\frac{g}{L}} \\
    \omega_S &= \sqrt{\frac{g}{L} + 2 \frac{k}{m}}
\end{align*}
\]
Beats

- Two oscillations of slightly different frequencies produce $\text{beats} = \text{modulation of amplitude}$
  - Coupled oscillators change their amplitudes by beats
- Beat frequency = difference of two frequencies
  - This is used in tuning piano, guitar, etc
- Will come back when we discuss group velocity
Finding Normal Modes

- Is there a **systematic way** of finding normal modes?
  - Symmetry is useful. But it does not always work
    - What if the two pendulums are different?
    - What if we coupled three pendulums?
  - We want a recipe that is guaranteed to work
- You need **linear algebra**
  - If you know linear algebra, this is an easy application
  - If you don’t, take it easy. This won’t hurt
## Cheat Sheet: Linear Algebra

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(M)</th>
<th>(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a_1, a_2, a_3))</td>
<td>((b_1, b_2, b_3))</td>
<td>(\begin{pmatrix} M_{11} &amp; M_{12} &amp; M_{13} \ M_{21} &amp; M_{22} &amp; M_{23} \ M_{31} &amp; M_{32} &amp; M_{33} \end{pmatrix})</td>
<td>(\begin{pmatrix} N_{11} &amp; N_{12} &amp; N_{13} \ N_{21} &amp; N_{22} &amp; N_{23} \ N_{31} &amp; N_{32} &amp; N_{33} \end{pmatrix})</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
a \pm b &= \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix} \\
ka &= \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \\
Ma &= \begin{pmatrix} M_{11}a_1 + M_{12}a_2 + M_{13}a_3 \\ M_{21}a_1 + M_{22}a_2 + M_{23}a_3 \\ M_{31}a_1 + M_{32}a_2 + M_{33}a_3 \end{pmatrix} \\
MN &= \begin{pmatrix} M_{11}N_{11} + M_{12}N_{21} + M_{13}N_{31} & M_{11}N_{11} + M_{12}N_{21} + M_{13}N_{31} & M_{11}N_{11} + M_{12}N_{21} + M_{13}N_{31} \\ M_{21}N_{11} + M_{22}N_{21} + M_{23}N_{31} & M_{21}N_{11} + M_{22}N_{21} + M_{23}N_{31} & M_{21}N_{11} + M_{22}N_{21} + M_{23}N_{31} \\ M_{31}N_{11} + M_{32}N_{21} + M_{33}N_{31} & M_{31}N_{11} + M_{32}N_{21} + M_{33}N_{31} & M_{31}N_{11} + M_{32}N_{21} + M_{33}N_{31} \end{pmatrix}
\end{align*}
\]
Rewriting Equation of Motion

We are looking for a normal mode

- Both \(x_1\) and \(x_2\) oscillate at one same frequency \(\omega\)

\[
\begin{align*}
\frac{d^2 x_1}{dt^2} &= -\frac{mg}{L} x_1 - k(x_1 - x_2) \\
\frac{d^2 x_2}{dt^2} &= -\frac{mg}{L} x_2 - k(x_2 - x_1)
\end{align*}
\]

\[
\begin{align*}
\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\frac{g}{L} & -\frac{k}{m} \\ \frac{k}{m} & -\frac{g}{L} - \frac{k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{align*}
\]

- Throw this into

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{i\omega t} \\ a_2 e^{i\omega t} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}
\]

\[
\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}
\]
Looking for a Normal Mode

\[-\omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} = \begin{pmatrix} \frac{-g}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{-g}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} \rightarrow -\omega^2 \mathbf{a} = \mathbf{K} \mathbf{a}\]

- Matrix \( \mathbf{K} \) times vector \( \mathbf{a} \) \( \rightarrow \) constant times \( \mathbf{a} \) itself
  - We call \( \mathbf{a} \) an eigenvector of \( \mathbf{K} \)
  - The constant \(-\omega^2\) is called the eigenvalue
- In general, an \( n \times n \) matrix has \( n \) eigenvectors
  - Barring unfortunate coincidence… Leave it to math
- We expect to find two normal modes here How?
Finding Eigenvalues

\(-\omega^2 a = Ka\)

\(\omega^2 a + Ka = 0\)

\(\begin{pmatrix}
\omega^2 - \frac{g}{L} - \frac{k}{m} & \frac{k}{m} \\
\frac{k}{m} & \omega^2 - \frac{g}{L} - \frac{k}{m}
\end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0\)

- Equation \(Ma = 0\) can be satisfied if \(\text{det}(M) = 0\).
  - The determinant in this case is
    \[
    \left(\omega^2 - \frac{g}{L} - \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = \left(\omega^2 - \frac{g}{L}\right)\left(\omega^2 - \frac{g}{L} - 2\frac{k}{m}\right) = 0
    \]
  - This gives the two solutions
    \[
    \omega_p = \sqrt{\frac{g}{L}}, \quad \omega_s = \sqrt{\frac{g}{L} + 2\frac{k}{m}}
    \]
Finding Eigenvectors

For $\omega = \omega_p$

\[ \begin{bmatrix} \omega^2 - \frac{g}{L} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \omega^2 - \frac{g}{L} - \frac{k}{m} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \]

$\omega_p = \sqrt{\frac{g}{L}}$, $\omega_s = \sqrt{\frac{g}{L} + 2 \frac{k}{m}}$

For $\omega = \omega_s$

\[ \begin{bmatrix} \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \]

\[ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix} \]
We found the eigenvalues and eigenvectors.

The normal modes are given by

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{i\omega t} \\ a_2 e^{i\omega t} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}
\]

\[
\begin{pmatrix} a \\ -a \end{pmatrix} e^{i\omega_{st} t}
\]
What Did We Learn?

- **Linear algebra** gives you a **recipe for finding normal modes** by solving an eigenvalue problem
  - We saw how it worked for our simple problem
- **It guarantees that the normal modes exist**
  - There will be $n$ normal modes if we couple $n$ pendulums
- We can now extend our problem knowing that the normal-mode, constant frequency solutions exist
  - This is **enough information** to let us attack the next problem – many many many coupled oscillators
Connect $N$ pendulums with springs

Displacement of the $n$-th pendulum is $x_n$ ($n = 1, 2, \ldots N$)

Equation of motion:

$$m \frac{d^2}{dt^2} x_n = -\frac{gm}{L} x_n - k(x_n - x_{n-1}) - k(x_n - x_{n+1})$$
Mass-Spring Transmission Line

\[ m \frac{d^2 x_n}{dt^2} = -\frac{gm}{L} x_n - k(x_n - x_{n-1}) - k(x_n - x_{n+1}) \]

- Assume that \( L \) is very long \( \rightarrow \) Ignore \( \frac{gm}{L} \)
  - The string just keep the mass from falling
  - Equivalent to mass-spring transmission line in H&L §4.2

\[ m \frac{d^2 x_n}{dt^2} = -k(x_n - x_{n-1}) - k(x_n - x_{n+1}) \]
Now we make $N$ very large, while making the mass and the spring smaller and smaller.

It starts to look like a spring with distributed mass.
- Good model for mechanical waves such as sound.
Summary

- Studied coupled oscillators
  - General solution is a linear combination of normal modes = patterns of oscillation with constant frequencies
  - Surprising pattern shows up – Beats
  - Linear algebra guarantees that the normal modes exist
    - Eigenvalues $\rightarrow$ Normal frequencies
    - Eigenvectors $\rightarrow$ Normal modes
- We are ready to extend coupled oscillators into mass-spring transmission line
- Real continuous waves next time!