Physics 15c Problem Set #5

Due on Friday, October 23, 4:00 PM

Problem 1
Calculate Fourier transform of the following functions.

(a) Gaussian function \( f(t) = \frac{1}{\sqrt{2\pi \sigma}} e^{\frac{-t^2}{2\sigma^2}} \), where \( \sigma \) is a positive real constant.

(b) Cusp function \( f(t) = e^{-\Gamma|t|} \), where \( \Gamma \) is a positive real constant.

Problem 2
Consider two functions \( f(t) \) and \( g(t) \), and their Fourier transform \( F(\omega) \) and \( G(\omega) \). The convolution of \( f(t) \) and \( g(t) \) is defined by

\[
h(t) = \int_{-\infty}^{\infty} f(s)g(t-s) \, ds.
\]

Show that the Fourier transform of the convolution is

\[
H(\omega) = 2\pi F(\omega) \cdot G(\omega).
\]

Problem 3
In Lecture #2, we saw that the transient solution for a weakly-damped oscillator was

\[
x(t) \approx e^{-\frac{\Gamma}{2} t} e^{-i\omega_0 t} \quad \text{if} \quad \Gamma \ll \omega_0.
\]

Compute the Fourier transform of \( x(t) \). (Note that the motion is non-zero only for \( t > 0 \).) Compare it with the motion of the driven oscillator from the same lecture.

Problem 4
In orchestral music, the parts for violins and flutes often have fast passages with short notes, while French horns and bassoons play more slowly. In piano music, fast melodies are always in the right hand (higher pitch). Why is this the case?

Suppose each note is a pure sinusoidal wave of frequency \( \nu \) [Hz] and duration \( T \) [seconds]. Because the duration is finite, the Fourier transform has a non-zero width. Let us assume that, in order to be recognized as a note of a certain pitch, the frequency spread of the note has to be smaller than 1/2 of a semitone. (A semitone is the frequency separation between adjacent keys, including black keys, of the piano. It’s about 6% of the frequency.) Find the relation between \( \nu \) and \( T \) that satisfies this condition. Calculate the shortest \( T \) for (a) the lowest note of the bassoon = 58 Hz, and (b) the highest note of the piccolo = 4186 Hz.
Problem 5
(a) Compact disks (CDs) record music by sampling the waveform at 44100 Hz with an accuracy of 16 bits per sample. In addition, there are two channels, one each for left and right ears. What is the total bandwidth, in bits per second?

(b) When you copy your CD into MP3 files, you can get “almost CD quality” sound with 128 kbps, or 128,000 Hz. What is the compression ratio? (If you are interested, research how such compression is possible.)

Problem 6
The phase velocity of surface waves in deep water is given by

\[
\frac{\omega}{k} = \sqrt{\frac{g + \frac{T_s}{k}}{\rho}}
\]

where \(g\) is the acceleration of gravity (9.8 m/s\(^2\)), \(T_s\) is the surface tension (7.3\(\times\)10\(^{-2}\) N/m) and \(\rho\) is the water mass density.

(a) Find the expression for the group velocity.

A stone thrown into a pond generate an impulse wave, much like Dirac’s \(\delta\)-function which includes components of all frequencies.

(b) What is the wavelength that travels at the slowest phase velocity?

(c) What is the wavelength for which the group velocity and the phase velocity are equal?

Problem 7
Electromagnetic waves traveling through a space filled with free electrons obey the following dispersion relation:

\[
k^2 = \frac{\omega^2}{c^2} \left( 1 - \frac{q^2 n_0}{\varepsilon_0 m \omega^2} \right),
\]

where \(q\) is the electron charge, \(n_0\) is the electron density, \(m\) is the electron mass. (We will derive this in Lecture #17.)

(a) What is the cut-off frequency below which electromagnetic waves cannot propagate through this medium?

(b) For frequencies above the cut-off, calculate the phase velocity \(c_p\) and the group velocity \(c_g\). Show that \(c_g\) does not exceed the speed of light \(c\) at any frequency \(\omega\).