Wave Phenomena

Physics 15c

Lecture 1
Waves
Harmonic Oscillator
Administravia

If you did not fill in the survey form last time, please do so
- The forms are in the back of the hall

Online sectioning is on
- Go to https://www.section.fas.harvard.edu/
- Both discussion sections and lab sections
- If the time slots do not work out for you, please send me email and explain your constraints

Talk to me if you haven’t taken E&M (15b/153)
Today’s Goals

Introduce the course topic: Waves
- What do we study, and why is it worthwhile

Lots of recap today
- Simple harmonic oscillators from 15a and 15b
- Complex exponential, Taylor expansion
- Make sure we all know the basics

Analyze simple harmonic oscillator using complex exp
- How do we interpret the complex solutions for a physical system?
- Are they general and complete?

Just how common are harmonic oscillators?
- Very – Physics is filled with them
- But why?
What we study in this course

There are waves everywhere

- Sea waves
- Sound
- Earthquakes
- Light
- Radio waves
- Microwave
- Human waves

The Great Waves off Kanagawa, Katsushika Hokusai, 1832
Features of waves

Oscillation at each space point
- Something ("medium") is moving back and force
  - Air, water, earth, electromagnetic field, people…

Propagation of oscillation
- Motion of one point causes the next point to move
  - How does oscillation propagate over distance?
  - What determines the propagation speed?

We study the general properties of waves focusing on the common underlying physics
Modern (\(= 20^{\text{th}}\) century) Physics has two pillars:

- **Relativity** was inspired by the absoluteness of the speed of light = electromagnetic waves
- **Quantum Mechanics** was inspired by the wave-like and particle-like behaviors of light

Everything is described by **wave functions**
- Relativistic QM is a theory of generalized waves

**Solid understanding of waves** is essential for studying advanced physics
Goal of This Course

Understand basic nature of wave phenomena
- Intuitive picture of how waves work
  - How things oscillate. How the oscillation propagates
  - How do waves transmit energy?
  - Why are waves so ubiquitous?

Foundation for more advanced subjects
- Familiarity with wave equations
- and Fourier transformation

Cover a few cool stuff related to waves
- Esp. electromagnetic waves
Simple Harmonic Oscillators

Already familiar with them, aren’t we?
Mass $m$ is placed on a friction-free floor

- Spring pulls/pushes $m$ with force
  \[ F = -kx \] (Hooke’s law)
- Newton’s law
  \[ F = ma = m \frac{d^2x}{dt^2} \]

We find the equation of motion:

\[ m \frac{d^2x}{dt^2} = -kx \]

- We must solve this differential equation for a given set of initial conditions
Equation of Motion

We know that the solution will look like a sine wave

- Try \( x = x_0 \cos \omega t \)

Equation of motion becomes

\[
m \frac{d}{dt} (x_0 \cos \omega t) = -kx_0 \cos \omega t
\]

\[
-mx_0 \omega^2 \cos \omega t = -kx_0 \cos \omega t
\]

We’ve found a solution

- Not necessarily \( \text{the} \) solution

Let’s remind ourselves how this solution looks like:

- How the position and the velocity change with time
- What is the frequency/period of the oscillation
- How the energy is (or is not) conserved

\[
\omega = \sqrt{\frac{k}{m}}
\]
Position, Velocity, Acceleration

\[ x = x_0 \cos \omega t \]
\[ v = \frac{dx}{dt} = -x_0 \omega \sin \omega t \]
\[ a = \frac{dv}{dt} = -x_0 \omega^2 \cos \omega t = -\omega^2 x \]

- Oscillation repeats itself at \( \omega t = 2\pi \)
- Position and velocity are off-phase by 90 degrees
  - Velocity is ahead
- Position and acceleration are off-phase by 180 degrees
\( \omega \) in \( \cos \omega t \) is the natural angular frequency of this oscillator
- How much the phase of the cosine advances per unit time
- Unit is [radians/sec]

The period \( T \) [sec/cycle] is given by

\[
2\pi = \omega T \quad \rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}
\]

The frequency \( \nu \) (Greek nu) [cycle/sec] is given by

\[
\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}
\]

\( \omega = \sqrt{\frac{k}{m}} \) a.k.a. Hertz
Energy

Spring stores energy when stretched/compressed:

\[ E_s = \frac{1}{2} kx^2 = \frac{1}{2} kx_0^2 \cos^2 \omega t \]

Moving mass has kinetic energy:

\[ E_k = \frac{1}{2} mv^2 = \frac{1}{2} mx_0^2 \omega^2 \sin^2 \omega t \]

\[ = \frac{1}{2} kx_0^2 \sin^2 \omega t \quad \text{Remember } \omega^2 = k/m \]

Therefore

\[ E_s + E_k = \frac{1}{2} kx_0^2 = \text{constant} \]
Energy Tossing

\[ E_S = \frac{1}{2} kx_0^2 \cos^2 \omega t \]

\[ E_K = \frac{1}{2} kx_0^2 \sin^2 \omega t \]

Energy moves between the spring and the mass, keeping the total constant.
We know both $\cos \omega t$ and $\sin \omega t$ are solutions

- The general solution is therefore
  \[ x(t) = a \cos \omega t + b \sin \omega t \]
  for arbitrary values of $a$ and $b$

It’s more convenient to use complex exponential $e^{i\omega t}$

- As we have learned in 15b/153

Next four slides are reminders on complex numbers
I assume you are familiar with complex numbers

- A few reminders to make sure we got the key concepts

\[ i = \sqrt{-1} \]

**Complex plane**

\[ z = a + ib \]

**Real part**

\[ \text{Re}(z) = a \]

\[ \text{Im}(z) = b \]

**Imaginary part**

\[ z^* = a - ib \]

**Complex conjugate**

\[ \text{Im}(z^*) = -\text{Im}(z) \]

\[ \text{Re}(z) = \frac{z + z^*}{2} \]

\[ \text{Im}(z) = \frac{z - z^*}{2i} \]
Absolute Value and Argument

For a complex number \( z \),

- The distance \( |z| \) from 0 is the **absolute value**:
  \[
  |z| = \sqrt{a^2 + b^2}
  \]

- The angle \( \theta \) is the **argument**, or **phase**:
  \[
  \theta = \text{arg}(z)
  \]

\( z \) may be expressed as:

\[
z = |z| (\cos \theta + i \sin \theta) = |z| e^{i\theta}
\]

using Euler’s identity

\[
e^{i\theta} = \cos \theta + i \sin \theta
\]
Euler’s Identity

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

This is a “natural” extension of the real exponential

- Check this with Taylor expansion

\[
e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \ldots
\]

\[
e^{ix} = 1 + ix - \frac{1}{2} x^2 - \frac{i}{6} x^3 + \frac{1}{24} x^4 + \frac{i}{120} x^5 + \ldots
\]

\[
\cos(x) = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \ldots
\]

\[
\sin(x) = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \ldots
\]
$e^{i\theta}$ goes around the unit circle on the complex plane.

$e^{i\theta} = \cos \theta + i \sin \theta$

http://xkcd.com/179/
Complex Solutions

Revisit the simple harmonic oscillator: \( \frac{d^2 x}{dt^2} = -\omega^2 x \)

- Substitute \( x = e^{\lambda t} \)

\[
\frac{d^2 e^{\lambda t}}{dt^2} = X^2 e^{\lambda t} = -\omega^2 e^{\lambda t} \quad \Rightarrow \quad X^2 = -\omega^2 \quad \Rightarrow \quad X = \pm i\omega
\]

We got two complex solutions to a harmonic oscillator

\[
x(t) = e^{\pm i\omega t}
\]

- They are complex conjugates of each other

- Generally, if you have a complex solution \( z(t) \) for an equation of motion, the complex conjugate \( z^*(t) \) must also be a solution

\[
\frac{d^2 z(t)}{dt^2} = -\omega^2 z(t) \quad \Rightarrow \quad \frac{d^2 z^*(t)}{dt^2} = -\omega^2 z^*(t)
\]
Since the equation of motion is linear, any linear combination of \( z(t) \) and \( z^*(t) \) is also a solution, i.e.,

\[
x(t) = \alpha z(t) + \beta z^*(t)
\]

where \( \alpha, \beta \) are complex constants.

Physical solution \( x(t) \) must be real.

\[
\text{Im}(x(t)) = \frac{x(t) + x^*(t)}{2i} = \frac{(\alpha z(t) + \beta z^*(t)) - (\alpha^* z^*(t) + \beta^* z(t))}{2i}
\]

\[
= \frac{(\alpha - \beta^*)z(t) - (\alpha^* - \beta)z^*(t)}{2i} = \text{Im}\left((\alpha - \beta^*)z(t)\right) = 0
\]

Therefore \( \alpha = \beta^* \), \( x(t) = \alpha z(t) + \alpha^* z^*(t) = 2\text{Re}(\alpha z(t)) \)

Ignoring the factor 2, this is the real part of \{ arbitrary complex number \( \alpha \) times one of the solutions \( z(t) \) \}
Generally, when \( z(t) \) and \( z^*(t) \) are complex solutions of an equation of motion, real (=physical) solutions are found by taking the real part of \( \alpha \cdot z(t) \), where \( \alpha \) is an arbitrary complex constant.

Going back to the harmonic oscillator:

- Expressing \( \alpha = a + ib \), we get

\[
x(t) = \text{Re} \left( (a - ib)e^{i\omega t} \right) = \text{Re} \left( (a - ib)(\cos \omega t + i \sin \omega t) \right)
\]

\[
= a \cos \omega t + b \sin \omega t
\]

- This is the general solution, as we knew from the beginning.

We will use this recipe throughout the course.
Ubiquity of Harmonic Oscillators

Harmonic oscillator’s equation of motion:

\[ m \frac{d^2 x}{dt^2} = -kx \]

The restoring force \(-kx\) is linear with \(x\)

- This is not exactly true in most cases
- Springs do not follow Hooke’s law beyond elastic limits

Still, the physical world is full of almost-harmonic oscillators
- And for a good reason
A pendulum swings because of the combined force of the gravity $mg$ and the string tension $T$

- Combined force is $mg \sin \theta$
- Displacement from the equilibrium is $L\theta$
- Force is not linear with displacement

A pendulum is not a harmonic oscillator

Taylor-expand $F = -mg \sin \theta$ around $\theta = 0$

$$
\sin \theta = \sin 0 + (\sin \theta)_{\theta=0}' \theta + \frac{1}{2} (\sin \theta)_{\theta=0}'' \theta^2 + \cdots
$$

$$
= \theta - \frac{1}{6} \theta^3 + \frac{1}{120} \theta^5 + \cdots
$$

- For small angle $\theta$,

$$
\frac{d^2 (L\theta)}{dt^2} = mL\ddot{\theta} = -mg\theta + O(\theta^3)
$$

Almost linear
Taylor Expansion

Any (smooth) function $f(x)$ can be approximated around a given point $x = a$ as:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \cdots + \frac{1}{n!}f^n(a)(x - a)^n + \cdots$$

- You are already familiar with this
- The approximation is better when $x - a$ is small
  - Because the higher-order terms $(x - a)^n$ shrinks faster
Look at the same problem with the potential energy

- At angle $\theta$, the mass $m$ is higher than the lowest position by $h = L(1 - \cos \theta)$
- The potential energy is $E_p = mgh = mgL(1 - \cos \theta)$

Taylor-expand $E_p$ around $\theta = 0$

$E_p = mgL(1 - \cos \theta) \approx \frac{1}{2} mgL\theta^2$

- Differentiating the energy by displacement gives you the force

$F = -\frac{dE_p}{dx} = -\frac{1}{L} \frac{d}{d\theta} \left( \frac{1}{2} mgL\theta^2 \right) = -mg\theta$

- OK, we got the linear force again…
We can often linearize the equation of motion for small oscillation around a stable point (equilibrium)

Why?
- Anything that is stable is at a minimum of the potential energy $E$
  - Let’s call it $x = 0$
- Taylor expansion of $E$ near $x = 0$ is
  $$E(x) = E(0) + E'(0)x + \frac{1}{2} E''(0)x^2 + \frac{1}{6} E'''(0)x^3 + \ldots$$
  - Since $x = 0$ is a local minimum, $E'(0) = 0$ and $E''(0) > 0$
  - For small oscillations, higher-order terms ($x^3, x^4, \ldots$) can be ignored
  $$E(x) \approx E(0) + \frac{1}{2} E''(0)x^2$$
  A simple parabola
Ubiquity of Harmonic Oscillators

- This gives a linear force
  \[ F = -\frac{dE}{dx} \approx -E''(0)x \]

Every physically stable object can make harmonic oscillation

- Stable object sits where the potential energy is minimum
- The potential near the minimum looks like a parabola
- Its derivative gives a linear restoring force

This is true for small oscillation

- How small depend on how the potential looks like
- We observe oscillation only when “small” is large enough

\[ E(x) \approx E(0) + \frac{1}{2} E''(0)x^2 \]
Analyzed a simple harmonic oscillator

- The equation of motion: \( m \frac{d^2x(t)}{dt^2} = -kx(t) \)
- The general solution: \( x(t) = a \cos \omega t + b \sin \omega t \) where \( \omega = \sqrt{\frac{k}{m}} \)

Studied the solution
- Frequency, period, energy conservation

Learned to deal with complex exponentials
- Makes it easy to solve linear differential equations

Studied how the equation of motion can be linearized for small oscillations
- Taylor expansion of the potential near the minimum

Next: damped and driven oscillators