Wave Phenomena
Physics 15c

Lecture 13
Spherical Waves
Doppler Effect, Shock Waves
What We Did Last Time

Waves in 2- and 3-dimensions

- Wave equation and normal modes easily extended

\[ \frac{\partial^2 \xi}{\partial t^2} = c_w^2 \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right) = c_w^2 \nabla^2 \xi \quad \xi = e^{i(k_x x + k_y y + k_z z - \omega t)} = e^{i(k \cdot x - \omega t)} \]

- Normal modes are plane waves
- Isotropy and (for EM waves) Lorentz invariance

Boundary conditions in 2-D and 3-D

- Rectangular drum, Chladni plate, sound in a room
- Natural extension of the 1-D problems such as a string
Goals For Today

Spherical Waves
- Multi-dimensional waves from a small source
- We know it spreads out – Exactly how?

Doppler Shift
- Waves generated by a moving source
- What if the observer (listener) is moving?

Shock Waves
- When the source moves faster than the waves
Waves From a Point Source

Plane waves were easy to handle

- But how do we make them?
  - We need, e.g., an infinitely large, flat speaker

Most waves are generated by a small object

- Size of the source $\ll$ Distance of transmission
  - Voice from a person’s mouth
  - Radio signals from a cellular phone
  - Light from the Sun
- We can approximate them as a point source

How do we describe waves spreading from a point?
Spherical Waves

Consider 3-D waves expanding from \( x = y = z = 0 \)

- Assume it is isotropic: \( \xi = \xi(r, t) \)
- i.e., it does not depend on the direction

Non-dispersive wave equation is

\[
\frac{\partial^2 \xi(r,t)}{\partial t^2} = c_w^2 \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right) = c_w^2 \nabla^2 \xi(r,t)
\]

- Express the Laplacian in spherical coordinates

\[
\frac{\partial^2 \xi(r,t)}{\partial t^2} = c_w^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \xi(r,t)}{\partial r} \right) + (\text{derivatives over } \theta \text{ and } \phi)
\]

\[
= c_w^2 \frac{1}{r} \left\{ 2 \frac{\partial \xi(r,t)}{\partial r} + r \frac{\partial^2 \xi(r,t)}{\partial r^2} \right\} = c_w^2 \frac{1}{r} \frac{\partial^2}{\partial r^2} \left( r \xi(r,t) \right)
\]
Wave Equation

The wave equation has become
\[
\frac{\partial^2}{\partial t^2} (r \xi(r,t)) = c_w^2 \frac{\partial^2}{\partial r^2} (r \xi(r,t))
\]

- The product \( r \cdot \xi(r,t) \) satisfies the usual 1D wave equation!

We know the solutions already

\[
r \xi(r,t) = A e^{i(kx \pm \omega t)} \quad \omega = c_w k \quad \Rightarrow \quad \xi(r,t) = A e^{i(kx \pm \omega t)} \frac{1}{r}
\]

- Sinusoidal waves with amplitude decreasing as \( 1/r \)
- Either expanding from or concentrating to the origin

If we are only interested in expanding waves

\[
\xi(r,t) = A e^{i(kx - \omega t)} \frac{1}{r}
\]
Wave Intensity

Intensity = energy flow density
- How much energy per unit area is being transmitted
- Always proportional to $(amplitude)^2$

For spherical waves, intensity falls off as $1/r^2$
- If you integrate over a sphere at $r = R$, total energy flow is

$$4\pi R^2 \times \frac{C}{R^2} = \text{const.}$$

Energy conservation
Spherical Waves vs. Normal Modes

Spherical waves and normal modes don’t look related

\[ \xi(r,t) = \frac{e^{i(kx \pm \omega t)}}{r} \leftrightarrow \xi(x,y,z,t) = e^{i(k \cdot x - \omega t)} \]

- Try adding up \( e^{i(k \cdot x - \omega t)} \) for a given \( |k| \)

\[
\begin{align*}
\int_0^\pi \int_0^{2\pi} e^{i(kr \cos \theta - \omega t)} \sin \theta d\theta d\phi &= 2\pi e^{-i\omega t} \int_{-1}^{+1} e^{ikrc} dc \\
&= 2\pi e^{-i\omega t} \left[ \frac{1}{ikr} e^{ikrc} \right]_{-1}^{+1} \\
&= 2\pi e^{-i\omega t} \left( \frac{1}{ikr} e^{ikr} - \frac{1}{ikr} e^{-ikr} \right) \\
&= \frac{2\pi}{ik} \left( \frac{e^{i(kr - \omega t)}}{r} - \frac{e^{i(-kr - \omega t)}}{r} \right)
\end{align*}
\]

Of course they are

Combination of expanding and shrinking waves
Our spherical waves are **isotropic**

- Real waves generated by small source are often directional
  - Speakers emit stronger sound forward than to the sides

Correct way of dealing with this is

\[ \xi = e^{-i\omega t} j_n(kr)Y_n(\theta, \phi) \]

- This gives a complete set of solutions in polar coordinates
  - \( n = 0 \) corresponds to our isotropic solution
  - You’ll learn this in QM for hydrogen atoms
In 1845, C.H.D. Buys Ballot did an experiment to test a prediction by Christian Doppler

- He hired a freight train and the trumpet section of the Vienna orchestra for two days
- Half of the players got on the train and played an Eb
- The other half did the same in the station
- Musicians could tell the difference between the notes
Moving Source

Source of the sound moves at $v_s$
- Original frequency is $f_0$
- Sound waves travels at $c_w$

We follow two peaks of the sound waves
- They are generated $T = 1/f_0$ apart
- Train moves $v_s T$ in the meantime

The distance between two rings is
- Arrival times at the listener differ by $(c_w \mp v_s)T$

$$
\begin{align*}
\frac{c_w \mp v_s}{c_w} T \\
\quad \text{Turn into frequency}
\end{align*}
$$

$$
f = \frac{c_w}{c_w \mp v_s} f_0
$$
Now we move the observer at $v_o$

- Sound source is at rest

Follow two peaks again $c_w T$

- Spacing between them is
- To the observer, sound seems to approach with velocity $c_w \mp v_o$
- Arrival times differ by

$$f = \frac{c_w \mp v_o}{c_w} f_0$$
Move both the source and the observer

- Combine the previous results
  \[ f = \frac{c_w \mp v_o}{c_w \mp v_s} f_0 \]

- Define direction of \( +v_s \) and \( +v_o \) as the direction of the sound
  \[ f = \frac{c_{\text{sound}} - v_{\text{observer}}}{c_{\text{sound}} - v_{\text{source}}} f_0 \]

Doppler shift for sound
Moving at an Angle

What if $v_s$ and $v_o$ are not parallel to the sound?

- Just take the component that is parallel to the sound

$$f = \frac{c_w - v_o \cos \theta_o}{c_w - v_s \cos \theta_s} f_0$$
Why does it matter who (source/observer) is moving?
- Wave (sound) velocity is constant relative to the medium
- Medium (air) gives an absolute reference of velocity

We must measure $v_s$ and $v_o$ relative to the air
- If there is wind (air is moving), you must add its effect
- Example: moving observer + constant wind

\[
f = \frac{c_w - v_o + v_w}{c_w + v_w} f_0
\]
Electromagnetic Waves

EM waves (light, radio waves, etc.) is special

1. There is no medium $\rightarrow$ No absolute reference
2. Whether source moves or observer moves shouldn’t matter
   - That’s what’s “relative” about Relativity

Suppose source and observer are approaching

- We can consider it in two ways

<table>
<thead>
<tr>
<th>Moving source</th>
<th>$V$</th>
<th>Stationary observer</th>
<th>Stationary source</th>
<th>$V$</th>
<th>Moving observer</th>
</tr>
</thead>
</table>

- The rule must be different…

$$ f = \frac{C}{C - V} f_0 $$

Not the same

$$ f = \frac{C + V}{C} f_0 $$
Relativistic Doppler Shift

Keyword: time dilation

Seen by the observer

- Source is moving at $v$
- Source’s clock runs slower by factor $1/\gamma$
- Which makes the frequency smaller by factor $1/\gamma$
- Multiply this to the usual Doppler shift

\[ f = \sqrt{1 - \frac{v^2}{c^2}} \frac{c}{c - v} f_0 = \sqrt{\frac{c + v}{c - v}} f_0 \]

Who (source/observer) is moving is irrelevant

Seen by the source

- Observer is moving at $v$
- Observer’s clock runs slower by factor $1/\gamma$
- Which makes him think that the frequency is larger by factor $\gamma$
- Multiply this to Doppler

\[ f = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{c + v}{c} f_0 = \sqrt{\frac{c + v}{c - v}} f_0 \]
Relativistic Doppler Shift

Time dilation occurs not only with light

- In principle, one should always take it into account
- In practice, it matters only when $v$ is very large
- Never an issue with sound (and other) waves

Time dilation does not depend on the direction of $v$

\[ f = \sqrt{1 - \frac{v^2}{c^2}} \frac{c}{c - v \cos \theta} f_0 = \frac{\sqrt{c^2 - V^2}}{c - v \cos \theta} f_0 \]

- For $\theta = 90^\circ$ \[ f = \sqrt{1 - \frac{v^2}{c^2}} f_0 \]

Time dilation occurs not only with light

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Time dilation does not depend on the direction of $v$
Expanding Universe

We know the Universe is expanding

- Distant stars are moving away from us
- Doppler shift makes their light lower in frequency
  - Longer in wavelength – toward red in visible spectrum
  - Called “redshift”

Use well-known spectral lines as references

- Lyman-\(\alpha\) line of hydrogen makes an ideal yardstick
Expanding Universe

Amount of redshift is expressed by

\[ z = \frac{\lambda}{\lambda_0} - 1 = \sqrt{\frac{c + v}{c - v}} - 1 \]

- Highest-redshift quasars and gamma-ray bursts reach \( z \approx 7–8 \)
- GRB 090423 had \( z = 8.2 \) and is \( 13 \times 10^9 \) light-years away
Shock Waves

What happens when a source moves at $c_w$?

- The source flies together with the sound waves it is making
- Waves pile up → Shock Wave

Energy accumulate at the wavefront
- Intensity $\rightarrow \infty$

Bullet flying at 1.01 $c_w$
Breaking the Sound Barrier

F/A-18 Hornet breaking the sound barrier

F-14 Tomcat breaking the sound barrier
Mach Cone

Once the object is faster than sound, the shock wave turns into a cone

- Opening angle $\theta$ given by
  \[ \sin \theta = \frac{c_w}{v_s} \]

Energy concentrates on the cone surface

- Where the cone touches the ground, observer hears a (super)sonic boom
- Another reason why supersonic aircrafts are unpopular
Čerenkov Effect

Can shock waves exist for light?
- Nothing travels faster than light in vacuum

In material, light slows down
\[ c \rightarrow \frac{c}{n} \]
- \( n = 1.33 \) for water
- Object with \( v > 0.75c \) makes shock waves of light in water
  - Called Čerenkov light
- β-ray electrons from nuclear reactors create “blue glow”

Reed Reactor generating 240 kW
Čerenkov light is often used for detection of high energy particles

- Example: DIRC detector in BABAR experiment
- Particles fly through rectangular bars of quartz ($n = 1.544$)
- Those with $v > 0.648c$ emit light
- Angle of light measured $\rightarrow$ Velocity of the particle
- Combine with momentum measurement $\rightarrow$ Mass
- Particle identification
How Well Does It Work?

pion ($m = 0.14 \text{ GeV/c}^2$)

kaon ($m = 0.49 \text{ GeV/c}^2$)
Summary

Studied spherical waves
- Wave equation of isotropic waves
  \[ \frac{\partial^2 (r \xi(r,t))}{\partial t^2} = c_w^2 \frac{\partial^2}{\partial r^2} (r \xi(r,t)) \]
- Solution \( e^{i(kx \pm \omega t)}/r \)
- Intensity decreases with \( 1/r^2 \)

Doppler shift
- Doppler shift for sound and for light
  - Special Relativity matters for light

Shock waves and Mach cones
- Object flying faster than sound
- Čerenkov effect for light in material

\[
\begin{align*}
\sin \theta &= \frac{c}{v} \\
\frac{f}{f_0} &= \frac{c_{\text{sound}} - v_{\text{observer}}}{c_{\text{sound}} - v_{\text{source}}} \\
\frac{f}{f_0} &= \sqrt{\frac{c + v}{c - v}}
\end{align*}
\]