Lecture 18

Interference
What We Did Last Time

Studied microscopic origin of refraction in matter
- Started from EM radiation due to accelerated charges
- Uniform distribution of such charges make EM waves appear to slow down
- Three levels of describing EM waves in matter
  - Point charges in vacuum
  - Current density in vacuum
  - Permittivity (or dielectric constant) of matter

Analyzed EM waves in plasma

\[ n_{\text{plasma}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \]

- Plasma reflective for \( \omega < \omega_p \)

\[ \omega_p = \sqrt{\frac{q^2 n_0}{\varepsilon_0 m}} \]
Insulators

Unlike in plasma or in metal, electrons in insulators are bound to the molecules

- Binding force is similar to a spring
- Rest of the molecule $M$ is much heavier → Ignore the movement

Equation of motion \( m\ddot{x} = qE_0 e^{-i\omega t} - k_s x \)

- Forced oscillation – We know the solution

\[ x = x_0 e^{-i\omega t} \quad \rightarrow \quad -m\omega^2 x_0 = qE_0 - k_s x_0 \quad \rightarrow \quad x_0 = \frac{qE_0}{k_s - m\omega^2} = \frac{qE_0}{m(\omega_0^2 - \omega^2)} \]

- Current density is

\[ J = qn_0 v = \frac{-i\omega q^2 n_0}{m(\omega_0^2 - \omega^2)} E_0 e^{-i\omega t} \]

\[ \omega_0 = \sqrt{\frac{k_s}{m}} \]
Dispersion Relation

Wave Equation is

\[ \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial J}{\partial t} \]

\[ -k^2 E = -\frac{\omega^2}{c^2} E - \frac{\mu_0 q^2 n_0}{m} \frac{\omega^2}{\omega_0^2 - \omega^2} E \]

- Dispersion relation is

\[ k^2 = \frac{\omega^2}{c^2} \left( 1 + \frac{q^2 n_0}{\varepsilon_0 m (\omega_0^2 - \omega^2)} \right) = \frac{\omega^2}{c^2} \left( 1 + \rho \frac{\omega_0^2}{\omega_0^2 - \omega^2} \right) \]

\[ \rho \equiv \frac{q^2 n_0}{\varepsilon_0 k_s} \]

- It’s dispersive again

- Phase velocity: \[ c_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 + \rho \omega_0^2 / (\omega_0^2 - \omega^2)}} \]

- Index of refraction: \[ n = \sqrt{1 + \rho \frac{\omega_0^2}{\omega_0^2 - \omega^2}} \]
Example: Air

Air is a mixture of N$_2$, O$_2$, H$_2$O, Ar, etc

- Many resonances exist in UV and shorter wavelengths
- Things are simpler in visible light, where $\omega \ll \omega_0$

\[
\frac{c}{\rho} = \frac{c}{\sqrt{1 + \rho \omega_0^2/(\omega_0^2 - \omega^2)}} \approx \frac{c}{\sqrt{1 + \rho}} \rightarrow n \approx \sqrt{1 + \rho} \approx 1 + \frac{1}{2} \rho
\]

We don’t really know $n_0$ and $k_s$, but we do know $n = 1.0003$ at STP $\Rightarrow \rho = 0.0006$

- $\rho$ is proportional to density $n_0$
- Using the ideal gas formula $PV = n_{\text{mol}}RT$

\[
n = 1 + 0.0003 \frac{273K}{T} \frac{P}{1\text{atm}} \quad \text{Index of refraction of air for low-freq. EM waves}$$

\rho \equiv \frac{q^2n_0}{\varepsilon_0 k_s}$
Mirage

In a hot day, air temperature is higher near the ground

\[ n = 1 + 0.0003 \frac{273K}{T} \frac{P}{1\text{ atm}} \]

- Light travels faster near the ground

Refraction makes light bend upward

- You may see the ground reflect light as if there is a patch of water
Goals For Today

Interference
- When waves overlap with each other, they strengthen or cancel each other

Thin-film interference
- Reflectivity of light through a thin transparent film
  - Newton’s rings
  - Anti-reflective coatings

Two-body interference
- Young’s experiment – Wave nature of light
Thin-Film Interference

We studied reflection at boundary between insulators

- Continuity of $E$ and $H$ gave us
  
  \[ E_R = \frac{n_1 - n_2}{n_1 + n_2} E_I \quad \quad E_T = \frac{2n_1}{n_1 + n_2} E_I \]
  
  \[ R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 \quad T = \frac{4n_1 n_2}{(n_1 + n_2)^2} \]

- Using $n$ instead of $Z$ assuming $\mu_1 = \mu_2$

What if we had 2 (or more) boundaries in series?

- For a pane of glass window, I said $T \to T^2$
- Things can get more interesting than that…
Two Boundaries

Consider transitions through two boundaries

- At \( z = 0 \),
  \[
  E_I + E_R = E_F + E_B \]
  \[
  \frac{E_I - E_R}{Z_1} = \frac{E_F - E_B}{Z_2} \]

- At \( z = d \)
  \[
  E_F e^{ik_2d} + E_B e^{-ik_2d} = E_T \]
  \[
  \frac{E_F e^{ik_2d} - E_B e^{-ik_2d}}{Z_2} = \frac{E_T}{Z_1} \]

- Note the exponentials

Solve for \( E_R, E_T, E_F, E_B \)

\[
E(z,t) = \begin{cases} 
  E_I e^{i(k_1z-\omega t)} + E_R e^{i(-k_1z-\omega t)} & \text{for } z \leq 0 \\
  E_F e^{i(k_2z-\omega t)} + E_B e^{i(-k_2z-\omega t)} & \text{for } 0 < z \leq d \\
  E_I e^{i(k_1(z-d)-\omega t)} & \text{for } z > d 
\end{cases}
\]
Two Boundaries

Solutions are

\[ E_R = \frac{(Z_1^2 - Z_2^2)(e^{ik_2d} - e^{-ik_2d})}{(Z_1 + Z_2)^2 e^{-ik_2d} - (Z_1 - Z_2)^2 e^{ik_2d}} \]

\[ E_T = \frac{4ZZ_1Z_2e^{-ik_2d}}{(Z_1 + Z_2)^2 e^{-ik_2d} - (Z_1 - Z_2)^2 e^{ik_2d}} \]

\[ E_F = \frac{2Z_2(Z_1 + Z_2)e^{-ik_2d}}{(Z_1 + Z_2)^2 e^{-ik_2d} - (Z_1 - Z_2)^2 e^{ik_2d}} \]

\[ E_B = \frac{2Z_2(Z_1 - Z_2)e^{ik_2d}}{(Z_1 + Z_2)^2 e^{-ik_2d} - (Z_1 - Z_2)^2 e^{ik_2d}} \]

- OK, so what did we learn?
Look at the reflectivity

\[
\frac{E_R}{E_I} = \frac{(Z_1^2 - Z_2^2)(e^{ik_2d} - e^{-ik_2d})}{(Z_1 + Z_2)^2 e^{-ik_2d} - (Z_1 - Z_2)^2 e^{ik_2d}}
\]

\[
R = \left| \frac{E_R}{E_I} \right|^2 = \frac{2(Z_1^2 - Z_2^2)^2 \sin^2(k_2d)}{8Z_1^2Z_2^2 + 2(Z_1^2 - Z_2^2)^2 \sin^2(k_2d)}
\]

- Reflectivity oscillates as a function of \( k_2d \)
- Reflection vanishes at \( k_2d = m\pi \) (\( m = 1,2,3,... \))
- What’s happening?

\[
Z_1 = 377\Omega, \ Z_2 = 250\Omega
\]
Path Lengths

What’s $k_2d = m\pi$?

- Wavelength in medium 2 is $\lambda_2 = \frac{2\pi}{k_2}$
- $k_2d = m\pi$ means $d = \frac{m\pi}{k_2} = m\frac{\lambda_2}{2}$
- $R = 0$ when $2d$ is a multiple of the wavelength

Consider two paths of reflection

- Lengths differ by $2d$
- Path B goes through an extra $e^{2ik_2d}$
  - Phase changes by $2k_2d$
- Reflections from A and B meet with the same phase if $2k_2d = m\times2\pi$

But then, shouldn’t they add up and increase $R$?
Hard and Soft Boundaries

Remember at a simple boundary

\[ E_R = \frac{n_1 - n_2}{n_1 + n_2} E_I \quad \quad E_T = \frac{2n_1}{n_1 + n_2} E_I \]

- **Direction of** \( E_R \) is
  - **Same** as \( E_I \) if \( n_1 > n_2 \), i.e. slow \( \rightarrow \) fast
  - Call it a **soft boundary**
  - **Opposite** to \( E_I \) if \( n_1 < n_2 \), i.e. fast \( \rightarrow \) slow
  - Call it a **hard boundary**

Reflection on a hard boundary flips the polarity of \( E \)

- Or, the **phase** of the reflected waves is **off by** \( \pi \)
- This is important when you are considering interference

\[ \varepsilon_1, \mu_1 \quad \quad \varepsilon_2, \mu_2 \]

\[ \begin{align*}
  H_I & \uparrow E_I & H_T & \uparrow E_T \\
  E_R & \uparrow H_R & E_R & \uparrow H_R \\
\end{align*} \]
Interference

Boundaries A/B are either hard/soft or soft/hard

- Two reflected waves have opposite polarities to start with

At $2k_2d = 2m\pi$, the paths A and B meet with opposite phase and cancel out

→ Destructive interference

How about $2k_2d = (2m - 1)\pi$?

- Boundaries cause a $\pi$ in phase difference
- Path difference gives another $\pi$

→ Constructive interference
Newton’s Rings

Place a convex lens on a glass plane

- Suppose $R_0$ is large $\rightarrow \theta$ is small
  $\rightarrow$ All surfaces are normal to the light
- Reflection depends on $2k_{\text{air}}d$

\[
d = R_0 - \sqrt{R_0^2 - r^2} \approx \frac{r^2}{2R_0} \quad \text{for } r \ll R_0
\]

\[
2k_{\text{air}}d = \frac{k_{\text{air}}r^2}{R_0} = \frac{2\pi r^2}{\lambda_{\text{air}} R_0}
\]

- Reflection vanishes at

\[
\frac{2\pi r^2}{\lambda_{\text{air}} R_0} = 2m\pi \quad \Rightarrow \quad r^2 = m\lambda_{\text{air}} R_0
\]
Newton’s Rings

Expect concentric rings

- Actual pattern less clear — and somewhat colored
- Different wavelengths make rings at different radii
- Ring pattern obscured after the first few

First maximum at \( 2k_{\text{air}}d = \pi \)
Back to Reflectivity

\[ R = \frac{2(Z_1^2 - Z_2^2)^2 \sin^2(k_2d)}{8Z_1Z_2^2 + 2(Z_1^2 - Z_2^2)^2 \sin^2(k_2d)} \]

Without interference, \( R \) would be \( 2 \times 4\% = 8\% \)

- Just adding the intensities of reflections (4% each) from two boundaries

Correct solution was obtained by:
- Adding the amplitudes of \( E \), and then
- Calculating the intensity = (amplitude)\(^2\)

Add amplitudes, NOT intensities

Air \( \rightarrow \) glass \( \rightarrow \) air

This is why reflectivity doesn’t add up
Destructive interference can be used to eliminate reflection on lenses

- Surface reflection of 4% each is unacceptable for lenses with many elements
- 23-element TV camera lens would lose 85% of the light

Idea: put a thin coating of material with intermediate $Z$

- Choose the thickness just right so that the reflection from two boundaries cancel each other
Now we have three materials

\[ n_{\text{air}} = n_1 < n_2 < n_3 = n_{\text{glass}} \]

- Both boundaries are "hard"
- No difference between reflection at A and B
- To eliminate reflection, path difference \( 2d \) must satisfy

\[
2k_2d = (2m - 1)\pi \quad \text{Odd multiple of } \pi \text{ makes A and B cancel}
\]

- Larger \( m \) won’t work well for white light
- Best solution: \( 2k_2d = \pi \), or \( d = \lambda_2 / 4 \)

Quarter wavelength coating
Coating Material

\( d = \frac{\lambda}{4} \) is good but not enough

- Need to match the amplitudes of reflection from A and B for perfect cancellation
- We know for each boundary

\[
\frac{E_R}{E_I} \bigg|_A = \frac{n_1 - n_2}{n_1 + n_2}, \quad \frac{E_R}{E_I} \bigg|_B = \frac{n_2 - n_3}{n_2 + n_3}
\]

- To make them equal, we need \( n_2 = \sqrt{n_1 n_3} \)

For air-to-glass, \( n_1 = 1.0003, \ n_3 = 1.5 \rightarrow n_2 = 1.22 \)

- Tune thickness for green (550 nm in vacuum)

\[
d = \frac{550}{4 \times 1.22} = 113 \text{ nm}
\]

Middle of visible spectrum
Human eyes most sensitive
Imagine two charged particles oscillating together
- Each radiate EM waves
- What’s the total radiation?

Very similar situation can be built by intercepting plane waves with a wall with two holes
- Waves going through the holes spread spherically
- Young’s double-slit experiment (1801–1803) demonstrated light was in fact made of waves
Geometrical Solution

What do we expect to see?
- Draw circular waves in red/blue around two sources
  - Represent peaks/valleys
- Look for crossings
  - Red/red or blue/blue $\rightarrow$ add up
    $\rightarrow$ Constructive interference
  - Red/blue $\rightarrow$ cancel each other
    $\rightarrow$ Destructive interference

Wave amplitude varies depending on direction
Two-Body Interference

E field generated by A and B differ in two ways

- Amplitude proportional to $1/x_1$ and $1/x_2$
  - This is negligible if $x_1 \approx x_2 \gg d$
- Phase differ by $k(x_1 - x_2)$

Ignore the amplitude difference

$$E_A = E_0 e^{i(kx_1 - \omega t)} \quad E_B = E_0 e^{i(kx_2 - \omega t)}$$

$$E = E_A + E_B = E_0 (e^{ikx_1} + e^{ikx_2})e^{-i\omega t}$$

$$= 2E_0 \cos\frac{k(x_1 - x_2)}{2} e^{i\left(\frac{k}{2}(x_1 + x_2) - \omega t\right)}$$

- Intensity goes with $S \propto \left(E_0 \cos\frac{k(x_1 - x_2)}{2}\right)^2$
- Need $x_1 - x_2$
Suppose $x_1 \approx x_2 \gg d$

- Two paths are almost parallel
- From geometry $x_2 - x_1 = d \sin \theta$

Intensity $I$ can be written as

$$I = I_0 \cos^2 \frac{k(x_1 - x_2)}{2} = I_0 \cos^2 \frac{kd \sin \theta}{2}$$

$$= I_0 \cos^2 \frac{\pi d \sin \theta}{\lambda}$$

- NB: still ignoring the $1/r$ dependence of amplitude
  - Exact solution much uglier than this
Intensity vs. Angle

- Approximate solution pretty good
- Result matches the guess from red/blue circles

\[ I = I_0 \cos^2 \frac{\pi d \sin \theta}{\lambda} \]

I used \( r = 5d \) for this plot
Difference invisible for larger \( r \)
In Young’s experiment, a screen was set up to observe the interference pattern. 

- It’s safe to ignore the $1/r$ dependence.
- Since $\theta$ is small, $\sin \theta \approx \tan \theta = y/D$.

$$I = I_0 \cos^2 \left( \frac{\pi d \sin \theta}{\lambda} \right) = I_0 \cos^2 \left( \frac{\pi dy}{\lambda D} \right)$$

- Simple $(\cos)^2$ pattern is found.
Young’s Experiment

We know light is electromagnetic waves
- It wasn’t so obvious in the 18th century
- Corpuscular model dominant in Newton’s era

Wavelength of light is too short
- Few wave-like phenomena can be observed in human-sized experiment (or daily experience)
- Young used interference to expand the wave-ness of light to a visible size

Story flips again in late 19th century
- Experiments (photoelectric effect, black body radiation) show particle-ness of light
- More about this later…
Summary

Studied interference
- >2 waves overlap
  → Amplitudes add up
  → Intensity = (amplitude)^2 does not add up

Thin-film interference
- Reflectivity of thin film depends on thickness, plus hard/soft-ness of the two boundaries
  - Newton’s rings
  - 1/4-wavelength anti-reflective coatings

Two-body interference
- Intensity depends on angle
  \[ I = I_0 \cos^2 \frac{\pi d \sin \theta}{\lambda} \]
- Young’s experiment demonstrated wave-ness of light