Wave Phenomena
Physics 15c

Lecture 2
Damped Oscillators
Driven Oscillators
What We Did Last Time

Analyzed a simple harmonic oscillator

- The equation of motion: \( m \frac{d^2 x(t)}{dt^2} = -kx(t) \)
- The general solution: 
  \[ x(t) = a \cos \omega t + b \sin \omega t \]
  where \( \omega = \sqrt{\frac{k}{m}} \)

Studied the solution
- Frequency, period, energy conservation

Learned to deal with complex exponentials
- Makes it easy to solve linear differential equations

Studied how the equation of motion can be linearized for small oscillations
- Taylor expansion of the potential near the minimum
Goals for Today

Damped oscillator
- Harmonic oscillator with friction = energy loss
- (Re)introduce quality factor $Q$

Driven (or “forced”) oscillator
- Resonance
- Good exercise of complex math

Finish discussion of single oscillators
Ideal $LC$ circuit is a harmonic oscillator $\rightarrow$ Real ones aren’t

- Because there is always resistance
- Oscillation dies away

Let’s analyze an RLC circuit

- **Step 1:** define current $I$
  - $I = \frac{dq}{dt}$
- ... and Charge $q$

- **Step 2:** Kirchhoff

\[
-L \frac{dl}{dt} - RI - \frac{q}{C} = 0 \quad \Rightarrow \quad \frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{q}{LC} = 0
\]

Easiest way to solve this: assume $q(t) = e^{xt}$

- Equation becomes $X^2 + \frac{R}{L} X + \frac{1}{LC} = 0$
RLC Circuit Solutions

\[ X^2 + \frac{R}{L}X + \frac{q}{LC} = 0 \]

solutions

\[ X = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \]

Three possible scenarios

- Positive \( \rightarrow \) 2 real solutions \( X = -\alpha \pm \beta \)
- Negative \( \rightarrow \) 2 complex solutions \( X = -\alpha \pm i\omega \)
- Zero \( \rightarrow \) 1 duplicate solution \( X = -\alpha \)

\[ \alpha = \frac{R}{2L}, \quad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}, \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]

Border line is \( R = 2\sqrt{\frac{L}{C}} \)

Smaller \( R \rightarrow \) complex solutions
Larger \( R \rightarrow \) real solutions
For $R < 2\sqrt{L/C}$, we find $q(t) = e^{-\alpha t} e^{\pm i\omega t}$

- Use Euler’s formula: $e^{ix} = \cos x + i \sin x$

  
  $q(t) = e^{-\alpha t} (\cos \omega t \pm i \sin \omega t)$

- Linear combinations of the solutions are

  
  $q(t) = e^{-\alpha t} (A \cos \omega t + B \sin \omega t)$

Note that $R \to 0$ recovers the simple $LC$ oscillator
Strong Damping

For \( R > 2\sqrt{\frac{L}{C}} \), we find \( q(t) = e^{-(\alpha \pm \beta)t} \)

- Note \( \alpha \pm \beta > 0 \), so both solutions are exponentially decreasing
- Linear combinations of the solutions are
  \[ q(t) = Ae^{-(\alpha + \beta)t} + Be^{-(\alpha - \beta)t} \]

\( A \) and \( B \) are determined by the initial conditions
- In our problem, \( q(0) = q_0, \quad i(0) = 0 \)

\[
q(0) = A + B = q_0 \\
i(0) = \left. \frac{dq}{dt} \right|_{t=0} = -(\alpha + \beta)A - (\alpha - \beta)B = 0
\]

\[
q(t) = q_0 \left( -\frac{\alpha - \beta}{2\beta} e^{-(\alpha + \beta)t} + \frac{\alpha + \beta}{2\beta} e^{-(\alpha - \beta)t} \right)
\]
Critical Damping

For $R = 2\sqrt{L/C}$, we find $q(t) = e^{-\alpha t}$ only

- We can’t satisfy the initial conditions

General solution turns out to be

$$q(t) = (A + Bt)e^{-\alpha t}$$

- Not trivial to come up with this
- In our problem, $q(0) = q_0$, $I(0) = 0$

$$q(0) = A = q_0 \quad I(0) = \frac{dq}{dt}\bigg|_{t=0} = -\alpha A + B = 0$$

$$q(t) = q_0 (1 + \alpha t)e^{-\alpha t}$$
For same $L$ and $C$, a critically damped oscillator loses energy faster than weakly/strongly damped oscillators.
How many times does a damped oscillator oscillate?

- For weak damping, the ratio \( Q \equiv \frac{\omega}{2\alpha} \) determines this

This is called the quality factor of the oscillator

- Defined as the number of radians it oscillates during the time it takes for the stored energy to decrease by a factor \( 1/e \)

- For an RLC circuit \( Q = \frac{\omega}{2\alpha} \approx \frac{1}{R \sqrt{LC}} \)

Large Q value = weak damping = more oscillations

Critical damping occurs at \( Q = 1/2 \)
Critical Damping

Critically damped oscillator stops most quickly

- The energy is dissipated in $R$ as fast as possible

This is useful when you are trying to control the movement

- Shock absorbers on automobiles
- Feedback control systems (e.g. thermostats)

Underdamped oscillation can cause disasters

- Overloaded tracks. Broken shock absorbers
Forced Oscillation

A motor drives one end of the spring
- \( \omega = \) Angular frequency of the motor

The spring is stretched/shrunk by
- \( r \cos \omega t \) on the motor side
- \( x(t) \) on the mass side

Equation of motion:
\[
F = -k(x(t) - r \cos \omega t)
\]

\[
m \frac{d^2 x(t)}{dt^2} + kx(t) = kr \cos \omega t
\]
Equation of Motion

\[ m \frac{d^2 x(t)}{dt^2} + kx(t) = kr \cos \omega t \]

We are interested in the motion caused by the motor
- The solution should oscillate with angular frequency \( \omega \)

Let's work in complex exponentials

\[
\cos \omega t \quad \text{replace} \quad e^{i\omega t}
\]

\[ x(t) = x_0 e^{i\omega t} \]

\[
(-m\omega^2 + k)x_0 e^{i\omega t} = kr e^{i\omega t}
\]

\[ x_0 = \frac{kr}{-m\omega^2 + k} \]

\[ x(t) = \frac{kr}{-m\omega^2 + k} e^{i\omega t} = r \frac{\omega_0^2}{\omega_0^2 - \omega^2} e^{i\omega t} \]

where \( \omega_0 = \sqrt{\frac{k}{m}} \)

- The amplitude of the oscillation depends on the driving frequency \( \omega \)

Natural frequency without the force
Friction

The oscillation amplitude \( x_0 \) goes to infinity when \( \omega = \omega_0 \)

- This is does not happen in reality because of friction

Include friction

Friction \( F \) is always opposite to the velocity \( v \)

Simplest model: \( F \) is proportional to \( v \)

- Not true in many cases
- Good enough approximation for today
- Must be OK if the velocity is small (Taylor expansion)

\[
F = -fv = -f \frac{dx(t)}{dt}
\]
Equation of Motion

\[ m \frac{d^2 x(t)}{dt^2} + f \frac{dx(t)}{dt} + kx(t) = kr \cos \omega t \]

We now have a friction term

- Let’s work with complex exponentials again

\[ (-m\omega^2 + i\omega + k)x_0 e^{i\omega t} = kr e^{i\omega t} \]

\[ x_0 = \frac{kr}{-m\omega^2 + i\omega + k} = \frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma \omega} \]

\( \Gamma \) represents the damping strength

- Corresponds to 2\( \alpha \) in the RLC circuit

- Quality factor is \( Q = \frac{\omega_0}{\Gamma} = \frac{\sqrt{mk}}{f} \)

\[ \begin{align*}
\omega_0 & \equiv \sqrt{\frac{k}{m}} \\
\Gamma & \equiv \frac{f}{m}
\end{align*} \]
Solution

\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} e^{i \omega t} \]

\[ \begin{align*}
\omega_0 & \equiv \sqrt{\frac{k}{m}} \\
\Gamma & \equiv \frac{f}{m}
\end{align*} \]

The **amplitude** and the **phase** of the oscillation depends on \( \omega \)
- We assume \( \Gamma \) is much smaller than \( \omega_0 \) (weakly damped)

At low frequency \( \omega \ll \omega_0 \)
\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} e^{i \omega t} \approx re^{i \omega t} \]
- The mass moves exactly following the motor

At high frequency \( \omega \gg \omega_0 \)
\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} e^{i \omega t} \approx -r \frac{\omega_0^2}{\omega^2} e^{i \omega t} \]
- The oscillation becomes small due to \( (\omega_0/\omega)^2 \)
Resonance

Suppose the motor is running at $\omega_0$

$$x(t) = \frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma \omega} e^{i\omega t} = \frac{r\omega_0}{i\Gamma} e^{i\omega t}$$

- The oscillation is larger than the movement of the motor end by the quality factor $Q = \frac{\omega_0}{\Gamma}$
  - The smaller the damping, the larger the oscillation
  - $Q$ quantifies how resonant a damped oscillator is
    - Large $Q \rightarrow$ Weakly-damped, highly-resonant
    - Small $Q \rightarrow$ Heavily-damped, muffled
- The phase is $90^\circ$ behind the driving force

$$x(t) = r \frac{\omega_0}{\Gamma} \sin \omega t$$
How They Look Like

$$Q = \frac{\omega_0}{\Gamma} = 3$$

$$\omega < \omega_0$$

$$\omega > \omega_0$$

$$\omega = \omega_0$$
Amplitude Near the Resonance

\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma \omega} e^{i\omega t} \]

The amplitude is

\[ |x| = \sqrt{xx^*} \]

\[ = \frac{r \omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + \Gamma^2 \omega^2}} \]

\[ Q = 10 \]

\[ Q = 3 \]
Phase Near the Resonance

The phase of the oscillation is

\[
\text{arg}\left(\frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega}\right) = -\text{arg}\left(\omega_0^2 - \omega^2 + i \Gamma \omega\right)
\]

\[
= \arccot\left(\frac{\omega^2 - \omega_0^2}{\Gamma \omega}\right)
\]

\[
\text{arg}(1/z) = -\text{arg}(z)
\]
The solution we found has no free parameter

- What happened to the 2 parameters in the initial condition?

Imagine we didn’t have the motor

- The equation of motion would be

\[ m \frac{d^2 x(t)}{dt^2} + f \frac{dx(t)}{dt} + kx(t) = 0 \]

- We know the general solution to it

\[ x(t) = x_0 e^{-\frac{\Gamma}{2} t} e^{i\omega_0 t} \]

\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma \omega} e^{i\omega t} \]
Add the “forced” and the “damped” solutions

\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} e^{i \omega t} + x_0 e^{-\frac{\Gamma}{2} t} e^{i \omega_0 t} \]

Because of linearity, the sum satisfies

\[ m \frac{d^2 x(t)}{dt^2} + f \frac{dx(t)}{dt} + kx(t) = re^{i \omega t} \]

This solution can satisfy any initial condition

- Second term disappears with time
- We concentrated on the first term ("steady-state solution")
What Happens to Energy?

Energy is not conserved because of the friction

- The energy consumed per unit time = (friction) x (speed)

\[
\begin{align*}
\text{friction} & = f\dot{x}(t) \\
\text{speed} & = \dot{x}(t)
\end{align*}
\]

Motor supplies the energy

- The work done per unit time = (force) x (speed) at the motor-spring connection

\[
\begin{align*}
\text{force} & = -k(x(t) - r \cos \omega t) \\
\text{speed} & = -r\omega \sin \omega t
\end{align*}
\]

\[ f \left\{ \dot{x}(t) \right\}^2 \quad kr \omega \sin \omega t(x(t) - r \cos \omega t) \]
Energy consumed by the friction must equal to the work done by the motor

- This is true only after averaging over time for one cycle
- Because the spring and the mass store/release energy

Next two slides show how to confirm $E_f = E_m$

- $E_f = $ energy consumed by the friction in one cycle
- $E_m = $ energy supplied by the motor in one cycle

Unfortunately, we must do this in real numbers, with sines and cosines

- Energy is not a linear quantity
Energy Consumed by Friction

\[ E_f = \int_0^T f \dot{x}^2 \, dt \]

\[ = f \int_0^T \Re \left( i\omega x_0 e^{i\omega t} \right)^2 \, dt \]

\[ = f \int_0^T \left( \frac{i\omega x_0 e^{i\omega t} - i\omega x_0^* e^{-i\omega t}}{2} \right)^2 \, dt \]

\[ = f \int_0^T \left( \frac{-\omega^2 x_0^2 e^{2i\omega t} + 2\omega^2 x_0 x_0^* - \omega^2 x_0^2 e^{-2i\omega t}}{4} \right) \, dt \]

\[ = \frac{f T \omega^2}{2} \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i\omega \Gamma} \frac{r \omega_0^2}{\omega_0^2 - \omega^2 - i\omega \Gamma} \]

\[ = \frac{f T r^2}{2} \frac{\omega_0^4 \omega^2}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ x(t) = x_0 e^{i\omega t} \]

\[ \Re(z) = \frac{z + z^*}{2} \]

\[ \int_0^T e^{\pm 2i\omega t} \, dt = 0 \]

\[ x_0 = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma \omega} \]
Work Done by Motor

\[ E_m = \int_0^T k(x(t) - r \cos \omega t)r \omega \sin \omega t \, dt \]

\[ = kr \omega \int_0^T \left( x_0 - r \right) e^{i \omega t} \left( x_0^* - r \right) e^{-i \omega t} \frac{e^{i \omega t} - e^{-i \omega t}}{2i} \, dt \]

\[ = kr \omega \int_0^T \left( x_0 - r \right) e^{2i \omega t} \left( x_0 - r \right) + \left( x_0^* - r \right) - \left( x_0^* - r \right) e^{-2i \omega t} \frac{4i}{4i} \, dt \]

\[ = -\frac{kr \omega T}{2} \text{Im}(x_0) = \frac{kr \omega T}{2} \frac{r \omega^2 \Gamma \omega}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ = \frac{f T r^2}{2} \frac{\omega_0^4 \omega^2}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ \Gamma = \frac{f}{m} \quad \frac{k}{m} = \omega_0^2 \]
Energy Balance Sheet

\[ \frac{E_f}{T} = \frac{E_m}{T} = \frac{f r^2}{2} \left( \frac{\omega_0^4 \omega^2}{\omega_0^2 - \omega^2} + \Gamma^2 \omega^2 \right) \]

So the intake equals the consumption on average.

Energy flow is proportional to \( f r^2 \)

- No energy is consumed if the friction \( f \) is zero

Low frequency limit

- \( \omega \ll \omega_0 \)

High frequency limit

- \( \omega \gg \omega_0 \)
On resonance

\[ \omega = \omega_0 \]

- The motor does large amount of work only when \( \omega \) is close to the resonance frequency \( \omega_0 \)
- The energy flow at the resonance is proportional to \( Q \)
  - The smaller the friction, the larger the loss?
  - Because the mass moves larger distance, and faster

\[ \frac{E_f}{T} = \frac{E_m}{T} = \frac{fr^2}{2} \frac{\omega^4}{\omega_0^2} \frac{\omega}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ \begin{align*}
\Gamma & \equiv \frac{f}{m} \\
Q & \equiv \frac{\omega_0}{\Gamma}
\end{align*} \]
Energy vs. Frequency

Average energy consumption for the same $m$ and $k$. 

\[
fr^2 \frac{\omega_0^4 \omega^2}{2 \left( \omega_0^2 - \omega^2 \right)^2 + \Gamma^2 \omega^2}
\]
Analyzed a damped oscillator
- Behavior depends on the damping strength
  - Weakly, strongly, or critically-damped
- How many oscillation does it do? → Q factor

Studied forced oscillation
- Oscillation becomes large near the resonance frequency
- Phase changes from $0 \rightarrow -\pi/2 \rightarrow -\pi$ as the frequency increases through the resonance
- Work done by the force is consumed by the friction
  - Energy consumption is large near the resonance
- Resonance is taller and narrower for a larger Q value

Next to come: coupled oscillators