What We Did Last Time

Analyzed a **damped oscillator**
- Behavior depends on the damping strength
  - Weakly, strongly, or critically-damped
- How many oscillation does it do? $\rightarrow$ Q factor

Studied **forced oscillation**
- Oscillation becomes large near the resonance frequency
- Phase changes from $0 \rightarrow -\pi/2 \rightarrow -\pi$ as the frequency increases through the resonance
- Resonance is taller and narrower for a larger Q value

Ran out of time $\rightarrow$ Continue today
Goals for Today

Wrap up the driven oscillator
- Initial conditions
- Power consumption and dissipation

Coupled Oscillators
- How a pair of harmonic oscillators behave when they are connected with each other.
- Will use some linear algebra

Prepare for real waves
The solution we found was

$$x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} e^{i \omega t}$$

- No free parameters! How can we satisfy the initial condition?

Imagine we didn’t have the motor

- The equation of motion would be
  $$m \frac{d^2 x(t)}{dt^2} + f \frac{dx(t)}{dt} + kx(t) = 0$$

- We know the general solution to it from Lecture #2
  - Assuming weak damping

$$x(t) = \exp \left( - \frac{f}{2m} + i \sqrt{k} - \frac{f^2}{4m^2} \right) t \right) \approx \exp \left( - \frac{\Gamma}{2} t \right) \exp \left( \omega_0 t \right)$$
Add the “forced” and the “damped” solutions

\[ x(t) = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} e^{i\omega t} + x_0 e^{-\frac{\Gamma}{2}t} e^{i\omega_0 t} \]

- The factor \( x_0 \) on the damped solution has two real parameters

- The sum satisfies \( m \frac{d^2 x(t)}{dt^2} + f \frac{dx(t)}{dt} + kx(t) = re^{i\omega t} \)

This solution can satisfy any initial condition

- Second term disappears with time
- We concentrated on the first term ("steady-state solution")
What Happens to Energy?

Energy is not conserved because of the friction

- The energy consumed per unit time = (friction) x (speed)

\[
\begin{align*}
\text{friction} &= f\dot{x}(t) \\
\text{speed} &= \dot{x}(t)
\end{align*}
\]

\[
f \left\{ \dot{x}(t) \right\}^2
\]

Motor supplies the energy

- The work done per unit time = (force) x (speed) at the motor-spring connection

\[
\begin{align*}
\text{force} &= -k(x(t) - r \cos \omega t) \\
\text{speed} &= -r \omega \sin \omega t
\end{align*}
\]

\[
kr \omega \sin \omega t(x(t) - r \cos \omega t)
\]
Energy consumed by the friction must equal to the work done by the motor

- This is true only after averaging over time for one cycle
- Because the spring and the mass store/release energy

Next two slides show how to confirm $E_f = E_m$

- $E_f = \text{energy consumed by the friction in one cycle}$
- $E_m = \text{energy supplied by the motor in one cycle}$

Unfortunately, we must do this in real numbers, with sines and cosines

- This is a bit of pain. Done in the next two slides
Energy Consumed by Friction

\[ E_f = \int_0^T f \dot{x}^2 \, dt \]

\[ = f \int_0^T \text{Re} \left( i \omega x_0 e^{i \omega t} \right)^2 \, dt \]

\[ = f \int_0^T \left( \frac{i \omega x_0 e^{i \omega t} - i \omega x_0^* e^{-i \omega t}}{2} \right)^2 \, dt \]

\[ = f \int_0^T \left( -\omega^2 x_0^2 e^{2i \omega t} + 2 \omega^2 x_0 x_0^* - \omega^2 x_0^2 e^{-2i \omega t} \right) \frac{4}{4} \, dt \]

\[ = fT \omega^2 \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \omega \Gamma} \frac{r \omega_0^2}{\omega_0^2 - \omega^2 - i \omega \Gamma} \]

\[ = \frac{fTr^2}{2} \frac{\omega_0^4 \omega^2}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ x(t) = x_0 e^{i \omega t} \]

\[ \text{Re}(z) = \frac{z + z^*}{2} \]

\[ \int_0^T e^{\pm 2i \omega t} \, dt = 0 \]

\[ x_0 = \frac{r \omega_0^2}{\omega_0^2 - \omega^2 + i \Gamma \omega} \]
Work Done by Motor

\[ E_m = \int_0^T k(x(t) - r \cos \omega t)r \omega \sin \omega t \, dt \]

\[ = kr \omega \int_0^T \text{Re} \left( (x_0 - r) e^{i\omega t} \right) \text{Im}(e^{i\omega t}) \, dt \]

\[ = kr \omega \int_0^T \frac{(x_0 - r) e^{i\omega t} + (x_0^* - r) e^{-i\omega t}}{2} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \, dt \]

\[ = kr \omega \int_0^T \left( x_0 - r \right) e^{2i\omega t} - (x_0 - r) + \left( x_0^* - r \right) - \left( x_0^* - r \right) e^{-2i\omega t} \frac{4i}{4i} \, dt \]

\[ = -\frac{kr \omega T}{2} \text{Im}(x_0) = \frac{kr \omega T}{2} \frac{r \omega^2 \Gamma \omega}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ = \frac{fTr^2}{2} \frac{\omega_0^4 \omega^2}{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

\[ \Gamma = \frac{f}{m} \quad k = \frac{\omega_0^2}{m} \]
Energy Balance Sheet

Average rate of energy intake equals consumption

\[ \frac{E_f}{T} = \frac{E_m}{T} = \frac{fr^2}{2} \frac{\omega_0^4 \omega^2}{\left( \omega_0^2 - \omega^2 \right)^2 + \Gamma^2 \omega^2} \]

Energy flow is proportional to \( fr^2 \)

- No energy is consumed if the friction \( f = 0 \)

Low frequency limit
- \( \omega << \omega_0 \)

\[ \approx \frac{fr^2 \omega_0^2}{2} \left( \frac{\omega}{\omega_0} \right)^2 \rightarrow 0 \]

High frequency limit
- \( \omega >> \omega_0 \)

\[ \approx \frac{fr^2 \omega_0^2}{2} \left( \frac{\omega_0}{\omega} \right)^2 \rightarrow 0 \]
Energy Balance Sheet

\[
\frac{E_f}{T} = \frac{E_m}{T} = \frac{fr^2}{2} \frac{\omega_0^4 \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2 \omega^2}
\]

On resonance

- \( \omega = \omega_0 \)
- The motor does large amount of work only when \( \omega \) is close to the resonance frequency \( \omega_0 \)
- The energy flow at the resonance is proportional to \( Q \)
- The full-width-at-half-maximum (FWHM) of the resonance peak is proportional to \( 1/Q \)

\[
\Gamma \equiv \frac{f}{m} \quad Q \equiv \frac{\omega_0}{\Gamma}
\]
Coupled Oscillators

Two identical pendulums are connected by a spring

- Consider small oscillation
  \[ x_1 \approx L\theta_1 \quad x_2 \approx L\theta_2 \]
- Spring tension is
  \[ F_S = -k(x_1 - x_2) \]
- Restoring force from gravity

\[ F_1 = mg \sin \theta_1 = \frac{mg}{L} x_1 \quad F_2 = \frac{mg}{L} x_2 \]

\[
\begin{align*}
    m \frac{d^2 x_1}{dt^2} &= - \frac{mg}{L} x_1 - k(x_1 - x_2) \\
    m \frac{d^2 x_2}{dt^2} &= - \frac{mg}{L} x_2 - k(x_2 - x_1)
\end{align*}
\]
The two equations are **symmetric**

By adding & subtracting them, you get

\[
\begin{align*}
    m \frac{d^2(x_1 + x_2)}{dt^2} &= -\frac{mg}{L} (x_1 + x_2) \\
m \frac{d^2(x_1 - x_2)}{dt^2} &= -\left(\frac{mg}{L} + 2k\right)(x_1 - x_2)
\end{align*}
\]

We can solve the equations for \((x_1 + x_2)\) and \((x_1 - x_2)\) separately, and find

\[
\begin{align*}
x_1 + x_2 &= Ae^{i\omega_P t} \\
x_1 - x_2 &= Be^{i\omega_S t}
\end{align*}
\]

\[
\begin{align*}
\omega_P &= \sqrt{\frac{g}{L}} \\
\omega_S &= \sqrt{\frac{g}{L} + 2\frac{k}{m}}
\end{align*}
\]
Parallel and Symmetric

\[
\begin{align*}
\begin{cases}
x_1 + x_2 &= Ae^{i\omega_P t} \\
x_1 - x_2 &= Be^{i\omega_S t}
\end{cases}
\quad \omega_P = \sqrt{\frac{g}{L}} \quad \omega_S = \sqrt{\frac{g}{L} + 2\frac{k}{m}}
\end{align*}
\]

Let \( B = 0 \rightarrow x_1 - x_2 = 0 \)

- Two pendulums are moving in parallel
- The spring does nothing
- \( \omega_P = \) natural frequency of free pendulum

Let \( A = 0 \rightarrow x_1 + x_2 = 0 \)

- Two pendulums are moving symmetrically
- The spring gets expanded/shrunk by twice the movement of each pendulum
- \( \omega_S \) is determined by both the pendulums and the spring
Normal Modes

The two oscillating patterns are called the normal modes

- Both are simple harmonic oscillation
  - Constant frequency & amplitude

Two normal modes for two coupled oscillators

- Two pendulums have two initial conditions each \((x_i \text{ and } dx_i/dt)\)
- Two normal modes have two parameters each \((a \cos \omega t + b \sin \omega t)\)
Initial Conditions

General solution is a linear combination of the normal modes

\[
\begin{align*}
    x_1 + x_2 &= Ae^{i\omega_p t} \\
    x_1 - x_2 &= Be^{i\omega_s t}
\end{align*}
\]

Let’s find a solution that satisfies an initial condition:

- \( x_1(0) = a, \dot{x}_1(0) = 0 \)
- \( x_2(0) = 0, \dot{x}_2(0) = 0 \)

\[\begin{align*}
    x_1 &= \frac{A}{2} e^{i\omega_p t} + \frac{B}{2} e^{i\omega_s t} \\
    x_2 &= \frac{A}{2} e^{i\omega_p t} - \frac{B}{2} e^{i\omega_s t}
\end{align*}\]

\[\begin{align*}
    A &= B = a
\end{align*}\]

Take the real parts, and remember some trigonometry

\[
\begin{align*}
    x_1 &= \frac{a}{2} (\cos \omega_p t + \cos \omega_s t) = a \cos \left( \frac{\omega_p + \omega_s}{2} t \right) \cos \left( \frac{\omega_p - \omega_s}{2} t \right) \\
    x_2 &= \frac{a}{2} (\cos \omega_p t - \cos \omega_s t) = -a \sin \left( \frac{\omega_p + \omega_s}{2} t \right) \sin \left( \frac{\omega_p - \omega_s}{2} t \right)
\end{align*}\]
Specific Solution

\[ \omega_S = 1.1 \times \omega_P \]
Specific Solution

\[
\begin{align*}
    x_1 &= \frac{a}{2} \cos \left( \frac{\omega_P + \omega_S}{2} t \right) \cos \left( \frac{\omega_P - \omega_S}{2} t \right) \\
    x_2 &= \frac{a}{2} \sin \left( \frac{\omega_P + \omega_S}{2} t \right) \sin \left( \frac{\omega_P - \omega_S}{2} t \right)
\end{align*}
\]

Plot shows the case where \( \omega_P \) and \( \omega_S \) are relatively close

\[ |\omega_P - \omega_S| \ll \omega_P + \omega_S \]

- The spring constant is small = weak coupling between the two pendulums

\[
\begin{align*}
    \omega_P &= \sqrt{\frac{g}{L}} \\
    \omega_S &= \sqrt{\frac{g}{L} + 2 \frac{k}{m}}
\end{align*}
\]

small
Beats

Two oscillations of slightly different frequencies produce beats = modulation of amplitude

- Coupled oscillators change their amplitudes by beats

Beat frequency = difference of two frequencies

- This is used in tuning piano, guitar, etc

Will come back when we discuss group velocity
Is there a **systematic way** of finding normal modes?

- **Symmetry** is useful. But it does not always work
  - What if the two pendulums are different?
  - What if we coupled three pendulums?
- We want a recipe that is guaranteed to work

You need **linear algebra**

- Relax. This is an easy application of what you already know
Rewriting Equation of Motion

Write the equations in a 2x2 matrix form

\[
\begin{align*}
    m \frac{d^2 x_1}{dt^2} &= - \frac{mg}{L} x_1 - k(x_1 - x_2) \\
    m \frac{d^2 x_2}{dt^2} &= - \frac{mg}{L} x_2 - k(x_2 - x_1)
\end{align*}
\]

\[
\begin{pmatrix}
    \frac{d^2}{dt^2} x_1 \\
    \frac{d^2}{dt^2} x_2
\end{pmatrix} =
\begin{pmatrix}
    - \frac{g}{L} & - \frac{k}{m} \\
    \frac{k}{m} & - \frac{g}{L} - \frac{k}{m}
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

- In normal modes, both \(x_1\) and \(x_2\) oscillate at one same frequency \(\omega\)

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} =
\begin{pmatrix}
    a_1 e^{i\omega t} \\
    a_2 e^{i\omega t}
\end{pmatrix} =
\begin{pmatrix}
    a_1 \\
    a_2
\end{pmatrix} e^{i\omega t}
\]

\[
\begin{pmatrix}
    \frac{d^2}{dt^2} x_1 \\
    \frac{d^2}{dt^2} x_2
\end{pmatrix} =
\begin{pmatrix}
    a_1 \\
    a_2
\end{pmatrix} e^{i\omega t}
\]

- Substitute these into the equation of motion
Looking for a Normal Mode

Matrix $K$ times vector $\mathbf{a}$ equals constant times $\mathbf{a}$ itself

- $\mathbf{a}$ is an eigenvector of $K$
- The constant $-\omega^2$ is the eigenvalue

In general, an $n \times n$ matrix has $n$ eigenvectors

- Barring unfortunate coincidence...

We expect to find two normal modes here
Finding Eigenvalues

\(-\omega^2 \mathbf{a} = \mathbf{K} \mathbf{a}\) leads to

\[ \mathbf{a}^T \begin{pmatrix} \omega^2 - \frac{g}{L} - \frac{k}{m} & k/m \\ k/m & \omega^2 - \frac{g}{L} - \frac{k}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \]

Equation \(\mathbf{M} \mathbf{a} = 0\) can be satisfied for a non-zero \(\mathbf{a}\) only if \(\det(\mathbf{M}) = 0\)

- The determinant in this case is

\[ \left( \omega^2 - \frac{g}{L} - \frac{k}{m} \right) - \left( \frac{k}{m} \right)^2 = \left( \omega^2 - \frac{g}{L} \right) \left( \omega^2 - \frac{g}{L} - 2 \frac{k}{m} \right) = 0 \]

- This gives the two solutions:

\[ \omega_P = \sqrt{\frac{g}{L}}, \quad \omega_S = \sqrt{\frac{g}{L} + 2 \frac{k}{m}} \]
Finding Eigenvectors

\[
\begin{pmatrix}
\omega^2 - \frac{g}{L} - \frac{k}{m} & \frac{k}{m} \\
\frac{k}{m} & \omega^2 - \frac{g}{L} - \frac{k}{m}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = 0 \quad \omega_P = \sqrt{\frac{g}{L}}, \quad \omega_S = \sqrt{\frac{g}{L} + \frac{2k}{m}}
\]

For \( \omega = \omega_P \)
\[
\begin{pmatrix}
-\frac{k}{m} & \frac{k}{m} \\
\frac{k}{m} & -\frac{k}{m}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}
\]

For \( \omega = \omega_S \)
\[
\begin{pmatrix}
\frac{k}{m} & \frac{k}{m} \\
\frac{k}{m} & \frac{k}{m}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix}
\]
We found the eigenvalues and eigenvectors.

The normal modes are given by

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  a_1 e^{i\omega t} \\
  a_2 e^{i\omega t}
\end{pmatrix}
= \begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix} e^{i\omega t}
\]

- For \( \omega_P \)

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  a \\
  a
\end{pmatrix} e^{i\omega_P t}
\]

- For \( \omega_S \)

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  a \\
  -a
\end{pmatrix} e^{i\omega_S t}
\]
What Did We Learn?

Linear algebra gives you a recipe for finding normal modes by solving an eigenvalue problem

- We saw how it worked for our simple problem

It guarantees that the normal modes exist

- There will be $n$ normal modes if we couple $n$ pendulums
- Subtlety: all eigenvalues $-\omega^2$ must be negative
  - This is true for any system near a stable equilibrium, because if there were a positive eigenvalue, it will give us a motion that expands exponentially with time

We can now extend our problem knowing that the normal-mode, constant frequency solutions exist

- This is enough information to let us attack the next problem – many many many many coupled oscillators
Many Coupled Pendulums

Connect $N$ pendulums with springs

- Displacement of the $n$-th pendulum is $x_n$ ($n = 1, 2, \ldots N$)
- Equation of motion:

\[ m \frac{d^2}{dt^2} x_n = -\frac{gm}{L} x_n - k(x_n - x_{n-1}) - k(x_n - x_{n+1}) \]

- Assume that $L$ is very long $\Rightarrow$ Ignore the $gm/L$ term

\[ m \frac{d^2}{dt^2} x_n = -k(x_n - x_{n-1}) - k(x_n - x_{n+1}) \]
Now we make $N$ very large, while making the mass and the spring smaller and smaller.

It starts to look like a spring with distributed mass.

- Good model for mechanical waves such as sound.
Summary

Studied coupled oscillators

- General solution is an linear combination of normal modes = patterns of oscillation with constant frequencies
- Surprising pattern shows up – Beats
- Linear algebra guarantees that the normal modes exist
  - Eigenvalues $\rightarrow$ Normal frequencies
  - Eigenvectors $\rightarrow$ Normal modes

We are ready to extend coupled oscillators into mass-spring transmission line

Next: continuous waves