Wave Phenomena
Physics 15c

Lecture 6
Sound
Transverse Waves
What We Did Last Time

Studied the energy and momentum carried by waves

- Energy is distributed non-uniformly over space
  - Kinetic energy = spring potential energy
- It travels at the wave velocity
- Momentum behaves similarly
  - Energy density / velocity = momentum density

Calculated what it takes to create the waves

- Power needed = energy transfer rate
- Force needed = momentum transfer rate

We know all about longitudinal waves now
Goals for Today

**Sound** in solid, liquid and gas
- Apply what we did in the last two lectures

**Transverse waves on string**
- Different kind of mechanical waves
- Define the wave equation and solve it
- Calculate velocity, energy density and momentum density
Sound

Sound is **longitudinal waves** in various material

- Medium can be solid, liquid or gas

Sound travels in 3 dimensions

- But much of the physics can be analyzed in 1 dimension

We already know everything about 1-dimensional longitudinal waves

- All we need to find out is the **linear mass density** $\rho_l$ and the **elastic modulus** $K$

This is going to be an easy lecture
A solid rod can be considered as a (very stiff) spring
- It can be compressed or stretched by force
- What is the elastic modulus $K$?

The elastic modulus is defined by $F = K \frac{\Delta l}{l}$

$K$ depends on the material and how thick the rod is
- It is proportional to the cross section of the rod
Young's Modulus

We can write $K = YA$

- $A$ is the cross section area (in $m^2$)
- $Y$ is Young's modulus of this material
  - Unit is Newtons/$m^2$

Hooke's law can be rewritten as $\frac{F}{A} = Y \frac{\Delta l}{l}$

- LHS: force per unit cross section
- RHS: Young’s modulus $\times$ relative deformation
- This is the microscopic law of elasticity, which is independent of the specific shape of the rod
Linear mass density \( \rho_l \) is easy to calculate
- Assume we know the volume mass density \( \rho_v \)
- The mass of the rod is \( \rho_v \times A \times l \)
- Divide by the length \( \rho_l = \rho_v \times A \)

Now we can calculate everything
- Just look up the formulas from the last lecture
Sound velocity in solid is

\[ c_w = \sqrt{\frac{K}{\rho_l}} = \sqrt{\frac{YA}{\rho_v A}} = \sqrt{\frac{Y}{\rho_v}} \]

- This is a material constant
- Example: steel has \( Y = 2 \times 10^{11} \text{ N/m}^2 \), \( \rho_v = 7800 \text{ kg/m}^3 \)

\[ c_w = \sqrt{\frac{2 \times 10^{11}}{7800}} = 5100 \text{ m/s} \]
Sound in Liquid

Liquid transmits sound the same way as solid
Consider a pipe filled with liquid

\[ F \Delta l = A \Delta V \]

- Force \( F \) changes the volume of a small section of the liquid

\[ l \rightarrow l + \Delta l \]

\[ V \rightarrow V + \Delta V \]

\[ \Delta V = A \Delta l \]
Elasticity of liquid is best described by the change of volume in response to pressure

- Liquid has no fixed shape → Can’t use “length” or “area”

\[
\text{Pressure } P = \frac{F}{A} = -\frac{M_B}{V} \frac{\Delta V}{V}
\]

- \( M_B \) is “bulk modulus” of the liquid
- We can calculate \( K \) from \( M_B \)

\[
F = -K \frac{\Delta l}{l} = -K \frac{\Delta V}{V} \quad \Rightarrow \quad K = M_B A
\]
Sound in Liquid

\[ \frac{F}{A} = -M_B \frac{\Delta V}{V} = -M_B \frac{\Delta l}{l} \]

- Elastic modulus \( K = M_B A \)
- Linear mass density \( \rho_l = \rho_v A \)

Wave velocity

\[ c_w = \sqrt{\frac{K}{\rho_l}} = \sqrt{\frac{M_B}{\rho_v}} \]

- Example: water

\[ c_w = \sqrt{\frac{M_B}{\rho_v}} = \sqrt{\frac{2.1 \times 10^9 \text{ N/m}^2}{10^3 \text{ kg/m}^3}} = 1.45 \times 10^3 \text{ m/s} \]
Sound in Gas

Sound in gas can be analyzed in the same way
- Gas is much more compressible, and much lighter

For gasses, we know a bit more about the constants
- Most gasses at normal $T$ & $P$ are very close to ideal gas
  - Ideal gas is made of infinitely small molecules that are not interacting with each other
- Properties of ideal gas is determined the molecular mass
  - $M_B$ and $\rho_v$ can be calculated
- No such luck with solid and liquid
  - $Y$, $M_B$, $\rho_v$ must be measured (or looked up in the tables)
1 mole of ideal gas occupies 22.4 liter under STP.

$$\rho_v = \frac{M_{\text{mol}}}{0.0224}$$

If not STP, use:

$$PV = nRT$$

- Pressure (N/m$^2$)
- Volume (m$^3$)
- Temperature (K)
- Gas constant 8.31 J/mol·K
- Amount (mol)

$$\rho_v = \frac{M_{\text{mol}}P}{RT}$$
Compressibility of Ideal Gas

How the pressure of gas reacts to compression?
- Ideal gas law $PV = nRT$ suggests $P \propto \frac{1}{V}$ \hspace{1cm} Wrong

Temperature goes up when gas is compressed
- This increases the pressure even more

$$P \propto \frac{1}{V^\gamma}$$  \hspace{1cm} $\gamma =$ ratio of specific heats $> 1$

- $\gamma = 5/3$ for monoatomic gasses (He, Ne, etc.)
- $\gamma = 7/5$ for diatomic gasses (H$_2$, N$_2$, O$_2$, etc.)
Bulk Modulus of Ideal Gas

The bulk modulus $M_B$ is defined by

$$\Delta P = -M_B \frac{\Delta V}{V} \quad \Rightarrow \quad M_B = -V \frac{dP}{dV}$$

- Using $P \propto \frac{1}{V^\gamma}$
  $$\frac{dP}{dV} = -\gamma \frac{P}{V} \quad \Rightarrow \quad M_B = \gamma P$$

We can now calculate sound velocity, etc., for any gas as long as it’s close to ideal gas
  - That is, it’s not too compressed or too cold

We just need to know the mass of 1 mole
- And $\gamma$…
Sound Velocity in Air

Air is about 80% N₂ + 20% O₂

1 mole weighs

\[ M_{\text{mol}} = 0.8 \times M_{N_2} + 0.2 \times M_{O_2} \]

\[ = 0.8 \times 28 \text{ g/mol} + 0.2 \times 32 \text{ g/mol} = 28.8 \text{ g/mol} \]

- \( \rho_v \) at STP is \( \rho_v = \frac{0.0288}{0.0224} = 1.29 \text{ kg/m}^3 \)

Both N₂ and O₂ are diatomic \( \rightarrow \) So is air

\[ M_B = \gamma P = \frac{7}{5} \text{ (1 atm)} = 1.4 \times 10^5 \text{ N/m}^2 \]

\[ c_w = \sqrt{\frac{M_B}{\rho_v}} = 330 \text{ m/s} \]
Sound Velocity in Air

If not at STP?

\[
M_B = \gamma P \\
\rho_v = \frac{M_{\text{mol}} P}{RT} \quad \Rightarrow \quad c_w = \sqrt{\frac{M_B}{\rho_v}} = \sqrt{\frac{\gamma PRT}{M_{\text{mol}} P}} = \sqrt{\frac{\gamma RT}{M_{\text{mol}}}}
\]

- \(M_{\text{mol}} = 0.0288 \text{ kg/m}^3, \quad \gamma = 7/5\) gives

\[
c_w = \sqrt{\frac{1.4 \times 8.31 \times T}{0.0288}} = 20.1 \sqrt{T} \text{ (K) m/s}
\]

Sound velocity of any (ideal) gas: \(c_w \propto \sqrt{T}\)

- It does **not** depend on the pressure \(P\)
- Next time you fly, check the outside temperature
  - If \(T = -60^\circ C\), \(c_w = 20\sqrt{273 - 60} = 292 \text{ m/s} = 1050 \text{ km/h}\)
Sound Intensity

Intensity of sound (how “loud”) is given by the energy carried by the sound

- Since sound can spread out, we need the density of energy per unit area

How much power (in Watts) goes through this area?
Human can hear sound in 20 Hz – 20 kHz
The intensity range is $10^{-12} \text{ W/m}^2 – 1 \text{ W/m}^2$
  - Normal conversation $\sim 10^{-6} \text{ W/m}^2$

We feel both frequency and intensity in log scale
  - Relative to 440 Hz (middle A)
    - 880 Hz is one octave higher
    - 1760 Hz is one octave higher again
  - Relative to $10^{-6} \text{ W/m}^2$
    - $10^{-5} \text{ W/m}^2$ is loud
    - $10^{-4} \text{ W/m}^2$ is twice as loud
What is the amplitude of 1 kHz sound wave with intensity $10^{-6}$ W/m$^2$?

- Imagine the sound is transmitted inside a pipe of cross-section $A$ m$^2$
- Density of air at STP is 1.29 kg/m$^3$ $\rightarrow \rho_i = 1.29A$
- Energy transfer rate is 

$$\frac{1}{2} c_w \rho_i \omega^2 \xi_0^2 = \frac{1}{2} (330)(1.29A)(1000 \times 2\pi)^2 \xi_0^2 = 10^{-6} A$$

$$\xi_0 = 1.1 \times 10^{-8} \text{ m}$$

That’s 0.01 microns

- Human ears are very sensitive
Where Are We Now?

Oscillators (harmonic, damped, forced, coupled)

Mechanical waves
- Transverse
- LC transmission line
- Electromagnetic waves

Longitudinal

Sound
- Velocity
- Energy
- Momentum

Waves

Energy

Momentum
Wave velocity, energy and momentum are the most important characteristics for all waves
- We have studied them only for longitudinal waves
- We must explore other types of waves

Study two other types of waves

We will then go on and study more interesting features of the wave phenomena
- e.g. reflection and standing waves
- We will use the 3 types of waves to illustrate them
A string is stretched by tension $T$
The string’s linear mass density is $\rho_l$
We vibrate the string vertically
- **Transverse** to the direction of the string
- Let’s call the displacement at $x$ as $\xi(x)$
Consider the piece between \( x \) and \( x + \Delta x \)

- Mass is \( m = \rho \Delta x \)

The tension at \( x \) has a slope

\[
\tan \theta = \frac{\partial \xi}{\partial x}
\]

- Assuming \( \theta \) is small, the vertical and horizontal components of the tension are

\[
-T \sin \theta \approx -T \frac{\partial \xi}{\partial x} \quad -T \cos \theta \approx -T
\]

Cancels between the two ends
Equation of Motion

Total vertical force is

\[-T \frac{\partial \xi(x)}{\partial x} + T \frac{\partial \xi(x + \Delta x)}{\partial x}\]

\[T \frac{\partial^2 \xi(x)}{\partial x^2} \Delta x\]

Equation of motion:

\[\rho I \Delta x \frac{\partial^2 \xi(x,t)}{\partial t^2} = T \frac{\partial^2 \xi(x,t)}{\partial x^2} \Delta x\]

- Looks identical to the longitudinal waves
- Solution must be the same

We had \(K\) here
Solution and Wave Velocity

The normal mode solutions should look

$$\xi(x,t) = \xi_0 \exp(i(kt \pm \omega t))$$

- Throwing it into

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = T \frac{\partial^2 \xi(x)}{\partial x^2}$$

we find

$$\rho l \omega^2 = Tk^2$$

- Wave velocity is

$$c_w = \frac{\omega}{k} = \sqrt{\frac{T}{\rho l}}$$

Tension (N)  
Mass density (kg/m)
Energy and Momentum

We can imagine that the energy and momentum densities are same as longitudinal waves with $K \rightarrow T$

\[
\text{Energy density } = \frac{1}{2} \rho \omega^2 \xi^2 \quad \text{Momentum density } = \frac{1}{2} \frac{\rho \omega^2 \xi_0^2}{c_w}
\]

- We must check these

First calculate how much power is needed to create transverse waves on the string
- Conservation of energy assures that this is identical to the energy transfer rate
- You will calculate the latter in the problem set
Creating Transverse Waves

To create \( \xi(x, t) = \xi_0 \cos(kx - \omega t) \), we drive the left end of the string by \( \xi_0 \cos \omega t \)

- What is the force against which we must work?

\[
T \cos \theta \approx T \\
T \sin \theta \approx -T \left( \frac{\partial \xi}{\partial x} \right)_{x=0}
\]
Power (Energy Transfer Rate)

The power is given by (force) x (velocity)

- Velocity is vertical
  \[ \frac{\partial (\xi_0 \cos \omega t)}{\partial t} = -\xi_0 \omega \sin \omega t \]

- Vertical component of the force is
  \[ -T \left( \frac{\partial (\xi_0 \cos(kx - \omega t))}{\partial x} \right)_{x=0} = -T \xi_0 k \sin \omega t \]

- Multiply them and take time average
  \[ T \xi_0^2 k \omega \sin^2 \omega t \xrightarrow{\text{Exactly as expected}} \frac{1}{2} T \xi_0^2 k \omega = \frac{1}{2} c_w \rho_l \omega^2 \xi_0^2 \]

\[ c_w = \frac{\omega}{k} = \sqrt{\frac{T}{\rho_l}} \]
Momentum Density

Can **transverse** waves carry **longitudinal** momentum?

- We need horizontal component of the movement
- It appears zero because we approximated $T \cos \theta \approx T$

The string do move horizontally

- Direction of motion is **orthogonal** to the string
- Vertical velocity $v_\perp = \frac{\partial \xi(x,t)}{\partial t}$
- Horizontal velocity $v_{||} = -v_\perp \tan \theta = -\frac{\partial \xi(x,t)}{\partial t} \frac{\partial \xi(x,t)}{\partial x}$
The momentum density is given by multiplying this with the mass density:

\[ \frac{dp}{dx} = \rho_l \omega k \xi_0^2 \sin^2(kx - \omega t) \]

Time average gives:

\[ \left\langle \frac{dp}{dx} \right\rangle = \frac{1}{2} \rho_l \omega k \xi_0^2 = \frac{1}{2} \frac{\rho_l \omega^2 \xi_0^2}{c_w} \]

Exactly as expected.

Energy and momentum carried by transverse waves are identical to those of longitudinal waves.
Summary

Studied **sound in solid, liquid and gas**
- Young’s modulus, bulk modulus and volume density determine everything
- **Ideal gas** is particularly simple: only need molecular mass

Analyzed **transverse waves on string**
- Wave equation looks the same as longitudinal waves
  - Just replace elastic modulus $K$ with tension $T$
- Solution has the same characteristics as well
- Energy and momentum densities are also identical

Next lecture: *LC transmission line*