Entanglement is at the heart of quantum mysteries and the power of quantum information. While 2-qubit entanglement is fairly well understood with good entanglement measures [1], understanding multiqubit entanglement remains a considerable challenge. With 3-qubits it is no longer enough simply to ask if the qubits are entangled; one must also ask \(j\) where entangled states are the maximally entangled to a third; the loss of any one of the qubits leaves the other two in a mixed state with only classical correlations. It is well known that it is no longer enough simply to ask if entanglement is fairly well understood with good entanglement measures such as logues of the GHZ to \(N\)-qubit states. This feature makes the 3-qubit state \(|\psi\rangle\), for converting the 3-qubit state \(|\psi\rangle\) into an approximate \(|\psi\rangle\) and characterize the change from the input to the output using quantum state tomography [12].

In the diagonal basis, \(|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\) and \(|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)\), we can rewrite \(|\psi\rangle = \frac{1}{2}(|\psi\rangle + |\psi\rangle + |\psi\rangle + |\psi\rangle)\). It is apparent in this basis that \(|\psi\rangle\) is a superposition of an unwanted term, \(|\psi\rangle\), and a \(W\) state. We define a local POVM with elements \(e_1 = |A\rangle\langle A| + a^2|D\rangle\langle D|\) and \(e_2 = (1 - a^2)\times|D\rangle\langle D|\), where \(a\) is a real number between 0 (perfect measurement) and \(1\) (no measurement). This POVM is applied to each photon in the state, and if all of the parties find \(e_1\), then the new state is \(|\psi\rangle = \mathcal{N}[a^3/2][|D\rangle + a\sqrt{3}/2|W\rangle]\), with the normalization constant, \(\mathcal{N} = 2/\sqrt{a^6 + 3a^2}\). The state \(|W\rangle = \frac{1}{\sqrt{2}}(|W\rangle + |W\rangle)\) is simply related to \(|W\rangle\) by three single-qubit rotations. It is clear from this state, that the unwanted term \(|\psi\rangle\) is reduced relative to the term \(|W\rangle\). It is also clear that one achieves a pure \(|W\rangle\) only in the limit as \(a \rightarrow 0\), where the probability of success also goes to zero. Nevertheless, the fidelity of this state with the desired \(|W\rangle\) is \(\mathcal{F}_W = \langle \psi | \langle \psi \rangle | \psi \rangle^2 = 3/(a^4 + 3)\), which rapidly rises from 3/4 to 1 as \(a\) decreases, i.e., as the strength of the measurement increases. Conversely, the fidelity with the GHZ state \(\mathcal{F}_{GHZ} = \langle \psi | \langle \psi \rangle | \psi \rangle^2 = (a^4 + 6a^2 + 9)/(4a^4 + 12)\) drops from 1 to 3/4 as \(a\) decreases.

Our method can be generalized to convert an \(N\)-qubit GHZ state \(|N\rangle\), where \(|N\rangle = \frac{1}{\sqrt{2}}(|H\rangle^{\otimes N} + |V\rangle^{\otimes N})\), into an arbitrarily good approximation to the \(N\)-qubit \(W\) state \(|W\rangle = \frac{1}{\sqrt{2}}(|W\rangle + |W\rangle)\). We use the fact that the GHZ states \(|N\rangle\) satisfy the following relation:

\[
|N\rangle = \frac{1}{\sqrt{2}}[|((N-M)+)|M\rangle + |((N-M)-)|M\rangle].
\]

where \(M < N\). Notice that this factorization preserves the
evenness or oddness in the number of negative signs. Through repeated application of these two rules, one can factor $|N+\rangle$ in terms of only single-qubit states $|D\rangle \equiv |1+\rangle$ and $|A\rangle \equiv |1-\rangle$. This reexpresses the GHZ state as an equally weighted superposition of all $2^{N-1}$ terms with an even number of $|A\rangle$’s. When $N$ is odd, the GHZ state can be directly rewritten as

$$|N+\rangle = \frac{1}{\sqrt{2^{N-1}}} [\sqrt{N} W_0^\phi + \sqrt{2^{N-1} - N} \phi],$$

where the state $\phi$ is a superposition of all those terms containing an odd number and at least 3 $|D\rangle$’s. When $N$ is even, application of a local transformation $|D\rangle \rightarrow |A\rangle$, $|A\rangle \rightarrow |D\rangle$ to any qubit allows the GHZ state to be written in the form of Eq. (2). If we apply the same local POVM to each of the $N$-qubits, and given that each POVM finds the element $e_{ij}$, then each unwanted amplitude is reduced by at least a factor of $a^2$ while each desired amplitude is reduced only by $a$. In general, the fidelity of the resultant state by this prescription with $|W_0^\phi\rangle$ is given by

$$F_N^{W_0} = \frac{2 a^2 N}{1 + a^2} \left(1 - a^2\right)^N,$$

regardless of whether $N$ is even or odd.

The details of our experimental method for creating 3-photon GHZ states can be found in [14]. In Fig. 1 ultraviolet laser pulses from a frequency-doubled Ti:Sapphire laser make two passes through a type-II phase-matched $\beta$-barium borate (BBO) crystal aligned, with walk-off compensation to produce 2-photon pairs, each in the Bell state $|\phi^+\rangle$ [15]. These two independent photon pairs can be further entangled when combined at the polarizing beam splitter (PBS1) and the four photons take four separate outputs $A$ and $B$. Recall that a PBS works by reflecting horizontally polarized light $|H\rangle$ and transmitting vertically polarized light $|V\rangle$. Thus, two photons that were incident from different sides can pass into different output modes only when their polarizations were both $|H\rangle$ or both $|V\rangle$. In this sense, the PBS acts as a quantum parity check [16].

Given that the parity check succeeds on the two photons from the independent pair, our state is transformed from the product state $|\phi^+\rangle_{12} |\phi^+\rangle_{34}$ to the 4-photon GHZ state $|4+\rangle = 1/\sqrt{2}(|HHHH\rangle_{AB14} + |VVVV\rangle_{AB14})$ [8]. We project photon 4 onto the state $|D\rangle$, and when this projection succeeds, it leaves $|\text{GHZ}\rangle = 1/\sqrt{2}(|HHH\rangle_{AB1} + |VVV\rangle_{AB1})$.

A tomographically complete set of measurements for a 3-photon polarization state requires 64 polarization measurements. We use the 64 combinations of the single-photon projections $|H\rangle$, $|V\rangle$, $|D\rangle$, and $|R\rangle$ on each of the three photons. These projections are implemented using a quarter-wave plate and polarizer for each of the photons $A$ and $B$, and a half- or quarter-wave plate and a fixed polarizer (actually a second PBS) for photon 1. Successful projections are signaled by 4-photon coincidence measurements, three photons for the state and one trigger photon, using single-photon counting single-photon counting detectors and coincidence logic. The most likely physical density matrix for our 3-qubits is extracted using maximum-likelihood reconstruction [17,18].

We begin with the GHZ state that was characterized previously via 3-photon quantum state tomography [14]. We rewrite the density matrix in the $|D/A\rangle$ basis; this gives the density matrix shown in Figs. 2(b) and 2(c) (real and
Polarized light is thus only 38% of that for the vertically polarized light. The transmission of the horizontally polarized light and 33% for the horizontally perfectly transmitted while the horizontal polarization is nonlocality and entanglement concentration of maximally partial polarizers have been used to study hidden nonlocality and entanglement concentration of maximally entangled mixed states [19]. We placed each such element so that, in the ideal case, the vertical polarization is perfectly transmitted while the horizontal polarization is partially reflected; we measured 88% transmission for the vertically polarized light and 33% for the horizontally polarized light. The transmission of the horizontally polarized light is thus only 38% of that for the vertically polarized light and we can describe the experiment using the POVM elements $\epsilon_1$ and $\epsilon_2$ with $a^2 = 0.38$. With this attenuation value, and beginning with the ideal GHZ state, the fidelity of the state with the ideal $W$ state given three POVM outcomes $\epsilon_1$ is expected to increase from 75% to 95%.

We used the same 64 tomographic measurement settings for the $W$ state as for the GHZ state. Data for each setting were accumulated for 1800 s and yielded a maximum of 120 fourfold coincidence counts (for the $|VVV\rangle$ projection). To account for laser power drift, which was small but not insignificant, we divided the fourfolds by the square of the singles at the trigger detector. Background fourfold coincidences from a twofold coincidence count and an uncorrelated accidental were estimated for each measurement setting and subtracted from the measured coincidences. Using the maximum-likelihood reconstruction, our most likely density matrix is shown in Figs. 3(b) and 3(c) (real and imaginary part, respectively). The density matrix for the ideal $W$ state in the $|HV\rangle$ basis consists of only nine real elements—three diagonals of 1/3 height corresponding to $|VVV\rangle$, $|VHV\rangle$, and $|VVH\rangle$ and six maximal positive coherences between them. It is clear from the data that the dominant elements in the density matrix are those same nine elements. In Fig. 3(a), we show the effect of the POVM with our experimentally measured attenuation on the ideal GHZ state. The diagonal elements are attenuated much more strongly, by $a^2$, while the coherences remain maximal and are thus reduced only by $a$. Note that one specific unwanted term contained in the

![FIG. 2 (color online). Density matrix of an ideal and experimentally measured 3-photon GHZ state. The reconstructed density matrix of (a) an ideal GHZ state (real part only) and our measured GHZ state, (b) real part and (c) imaginary part, from Ref. [14]. The state is displayed in the $D/A$ basis, where $D = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $A = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$ are defined as in the text. The grid plots display the absolute value of each component of the matrices and are meant to show the structure of the matrix, namely, that our GHZ state is characterized in this basis by four diagonal elements of equal height with maximal positive coherences. The experimentally measured density matrix has a fidelity of 77% with the ideal GHZ state and 79% with any state related to the ideal GHZ state via local unitary transformations.](image-a)

![FIG. 3 (color online). Density matrix of the approximate W output state after the POVM procedure. The reconstructed density matrix of our output state, (b) real part, and (c) imaginary part, after the three local POVM operations. The application of the POVMs have suppressed several of the matrix components such that the final state contains only nine major elements. These are the same nine elements for the ideal $W$ state. The operation has increased the fidelity of our state with a $W$ state from 61% to 68%, while at the same time reducing the fidelity of our state with a GHZ state from 79% to 60%. For comparison, we show (a) the action of the POVM operation on the ideal GHZ state. Although this state still contains large coherences with the $HHH$ component, this state has 95% fidelity with the $W$ state.](image-b)
diagonal element for $|V V V\rangle$ is much more significant after the application of the POVM; this noise contribution, in the ideal case, is untouched by the POVM, and in the real case, least reduced.

We characterize the changes in our states using fidelity. The fidelity of a density matrix $\rho$ with a pure quantum state $|\psi\rangle$ is given by $F = \langle \psi | \rho | \psi \rangle$. We calculate this fidelity with a general GHZ ($W$) state, $|\text{GHZ}_G\rangle$ ($|W_G\rangle$), which is related to $|\text{GHZ}\rangle$ ($|W\rangle$) by three local unitary rotations. The initial state has a fidelity of $F_{\text{GHZ}_G} = (79.4 \pm 1.6)\%$ and $F_{\text{W}_G} = (60.5 \pm 1.9)\%$ as compared with the ideal 100% and 75%. After successful application of three local POVMs, $F_{\text{GHZ}_G} = (59.8 \pm 2.5)\%$ and $F_{\text{W}_G} = (68.4 \pm 2.4)\%$. Uncertainties in quantities extracted from these density matrices were calculated using a Monte Carlo routine and assumed Poissonian errors. A theoretical calculation based on our measured initial state and measured $a$ has yielded a final state with $F_{\text{W}_G} = 75\%$. Thus much of the difference with the expected fidelity in the ideal case is a result of the quality of the initial state. Nevertheless, the overlap with a $W$ state has been significantly improved, while the overlap with a GHZ state has been strongly reduced.

We have described a method for converting $N$-qubit GHZ states to arbitrarily good approximations to $N$-qubit $W$ states based on generalized quantum measurements or POVMs. We have implemented this scheme for the 3-qubit case and characterized the input and output states using multiphoton quantum state tomography. We have quantitatively shown that the transformation induced by the generalized measurement realized using partial polarizers results in a decrease in the overlap of the state with a GHZ state while increasing the overlap with a $W$ state. Multiparticle entanglement is essential to the success of quantum information processing. The theory and experimental work presented here extends our abilities to manipulate and understand the relationship between different types of complex entangled states.

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