Schramm-Loewner Evolution

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History of SLE

• 1923 - Charles Loewner used the Loewner equation in his proof of a special case of the Bieberbach conjecture

• 1999 - Oded Schramm used Brownian motion as input, proved the Mandelbrot four-thirds conjecture
Applications of SLE

• Highly applicable to physics
  – Liouville quantum gravity
  – Ising model for the motion of atoms
• Loop-erased random walks and random Peano curves
The Loewner Equation

Chordal:
\[ \dot{z} = \frac{2}{z - \lambda(t)} \]

Radial:
\[ \dot{z} = -z \frac{z + \lambda(t)}{z - \lambda(t)} \]
Chordal vs. Radial SLE

- Domain of chordal SLE: upper halfplane
- Domain of radial SLE: unit disc
- Unit disc and upper halfplane are conformally equivalent
- So, results for one are applicable to the other
- We focus on chordal:

\[
\dot{z} = \frac{2}{z - \lambda(t)}
\]
Basics of Chordal SLE

Separating into real and imaginary:

\[ \dot{z} = 2 \frac{x - \lambda(t) - iy}{(x - \lambda(t))^2 + y^2} \]

\[ \dot{x} = 2 \frac{x - \lambda(t)}{(x - \lambda(t))^2 + y^2} \]

\[ \dot{y} = 2 \frac{-y}{(x - \lambda(t))^2 + y^2} \]
The Domain of Singularities

- The most studied and most important aspect of SLE.
- Hull: the set of all points that crash into the driving function
- Tip: the curve that generates the hull
Brownian Motion

• Random walk: think of flipping a coin: heads move up one space, tails move down one space

• Brownian motion is limit as step size goes to 0, and it is scale invariant

• $\kappa$ denotes the speed, $B_{\kappa}(t) \sim N(0, \kappa t)$
Special Values of $\kappa$

- For $\kappa \leq 4$, the tip is almost surely a simple curve (doesn’t intersect itself) and only intersects the real axis at $t=0$.
- For $4 < \kappa < 8$, the tip almost surely intersects itself and the real axis.
- For $\kappa \geq 8$, the tip is space-filling.
- The tip has Hausdorff dimension $1 + \kappa/8$ for $\kappa \leq 8$ and dimension 2 for $\kappa > 8$. 
A Related Result

- Let $\lambda$ be Lip($\frac{1}{2}$) with $\lambda(0) = 0$ and let $x_0 > 0$. Suppose that $x(t)$ is a solution to the Loewner equation and that $x(1) = \lambda(1)$. Then $\| \lambda \|_{\frac{1}{2}} \geq 4$. (Joan R. Lind)
A Potential Link

- Let $A_c = \{ \exists t \in [0, 1] : (B(t + h) - B(t)) > c\sqrt{h} \forall h \in (0, 1) \}$
- Then $P(A_c) = 0$ if $c > 1$ and $P(A_c) > 0$ if $c < 1$.
- This scales with $\kappa^{1/2}$
Funnels and Fences
A Construction Project

Vector Field for the Loewner equation:

\[ f(x, y, t) = \frac{-2(\lambda(t) - x) - 2iy}{(\lambda(t) - x)^2 + y^2} \]

Equation for the upper fence:

\[ y - k(\lambda(t) - x) = 0 \]

Taking the dot product:

\[ -2k(\lambda(t) - x) - 2y - k\lambda'(t)((\lambda(t) - x)^2 + y^2) < 0 \]

Corresponding vectors:

\[
\begin{bmatrix}
-2(\lambda(t) - x) \\
-2y \\
(\lambda(t) - x)^2 + y^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{k}{1} \\
-\frac{k\lambda'(t)}{-} \n\end{bmatrix}
\]
Substituting the constraint for $y$, canceling common factors:

$$-4 - \lambda'(t)(\lambda(t) - x)(1 + k^2) < 0$$

Plugging in a specific driving function:

$$\lambda(t) = a\sqrt{|\cos(t)|} \quad \lambda'(t) = \frac{-asin(t)}{2\sqrt{|\cos(t)|}}$$

Setting:

$$(\lambda(t) - x) = c\lambda(t) \quad \sin(t) = 1$$

Yields:

$$a^2c(1 + k^2) < 8$$

So, given any $a$, there exists $k$ and $c$ small such that we can construct an upper fence.
Vertical wall for the inner fence:

\[ x = b \sqrt{|\cos(t)|} \]

Producing the inequality:

\[ 4(a - b) \sqrt{|\cos(t)|} - b \frac{\sin(t)}{\sqrt{|\cos(t)|}} ((a - b)^2 |\cos(t)| + y^2) < 0 \]

Setting \( y = 0 \), cancelling common factors:

\[ 4(a - b) < b(a - b)^2 \quad \text{So,} \quad 4 < b(a - b) \]

And, \( a > 4 \)
Checking that the fences meet:

\[ 4 < b(a - b) \quad a^2 c(1 + k^2) < 8 \quad c = \frac{a - b}{a} \]

Leads to:

\[ \frac{a + \sqrt{a^2 - 16}}{2} > b > \frac{a^2 - 8}{a} \]

And:

\[ a > 4 \]
The Hull

\[ \lambda(t) = \sqrt{|\cos(t)|} \]

\[ \lambda(t) = 5 \sqrt{|\cos(t)|} \]
The Brownian Case

- Can’t build upper fence along driving function because of non-differentiability
- Can still construct an inner fence for $\kappa > 16$
- Implies a two-dimensional hull
Vertical wall for the inner fence:

\[ x = b\sqrt{d-t} \]

Vector field produced by Brownian motion:

\[
\begin{bmatrix}
-1 \\
0 \\
-b \\
2\sqrt{d-t}
\end{bmatrix}
\]

Yielding the inequality:

\[ 4(B(t) - x) - \frac{b}{\sqrt{d-t}}((B(t) - x)^2 + y^2) < 0 \]
Setting $y=0$, cancelling common terms:

$$4 < \frac{b}{\sqrt{d-t}}(B(t) - x)$$

Recall:

$$B(t) - x = B(t) - b\sqrt{d-t} > (\sqrt{\kappa} - b)\sqrt{d-t}$$

Yielding:

$$4 < b(\sqrt{\kappa} - b)$$

$$\kappa > 16$$
Completing the Proof

• For $\kappa > 16$, it is possible to create a funnel on the real axis.
• So, there exists an interval of points along the real axis that crash into the driving function.
• Thus, the tip intersects the real axis.
• And the hull, bounded by the tip and the real axis, has two-dimensional area.
Why Not $\kappa > 4$?

• “Notice that according to the law of the iterated logarithm, a.s. Brownian motion is not Holder continuous with exponent 1/2. Therefore, it seems unlikely that the results of the present paper can be obtained from deterministic results.” –Rohde and Schramm

• Law of iterated logarithm:

$$\limsup_{\varepsilon \to 0^+} \frac{|B(\varepsilon)|}{\sqrt{2\varepsilon \log(\log(1/\varepsilon))}} = 1$$
$k = 1$
κ = 4
κ = 5
$k = 8$

Pictures by Tom Kennedy
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References


THANK YOU