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Lecture 10:
Energy and Power in Waves

1 Energy in a string

The kinetic energy of a mass m with velocity v is $\frac{1}{2}mv^2$. Thus if we have a oscillating wave in a string, the kinetic energy of each individual bit of the string is

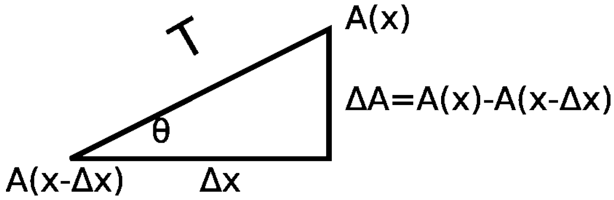
$$\mathrm{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(\mu\Delta x)\left(\frac{\partial A(x,t)}{\partial t}\right)^2 \tag{1}$$

Thus the kinetic energy per unit length is

$$\frac{\mathrm{KE}}{\text{length}} = \frac{1}{2}\mu\left(\frac{\partial A(x,t)}{\partial t}\right)^2 \tag{2}$$

The potential energy depends on how stretched the string is. Of course, having a string with some tension T automatically has some potential energy due to the stretching, even if there are no waves passing through the string. We are instead interested in the potential energy stored in the string as it is stretched further, due to the propagation of transverse waves. For simplicity, we use potential energy to refer to only this additional potential energy due to the extra strength.

If the string is in equilibrium, so $\frac{\partial A}{\partial x} = 0$, then by definition the potential energy is zero. The amount the string is stretched at point x is given by the difference between the length of the hypotenuse of the triangle and the base. Recall our triangle:



The amount the string is stretched is

$$\Delta L = \sqrt{(\Delta x)^2 + [A(x) - A(x-\Delta x)]^2} - \Delta x \tag{3}$$

$$= \Delta x\sqrt{1 + \left(\frac{\Delta A}{\Delta x}\right)^2} - \Delta x \tag{4}$$

Since the string is close to equilibrium $\frac{\Delta A}{\Delta x} = \frac{\partial A}{\partial x} \ll 1$, so we can Taylor expanding the square-root

$$\Delta L = \Delta x\left(1 + \frac{1}{2}\left(\frac{\partial A}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial A}{\partial x}\right)^4 + \cdots\right) - \Delta x \tag{5}$$

and drop subleading terms

$$\Delta L = \frac{1}{2}\Delta x\left(\frac{\partial A}{\partial x}\right)^2 \tag{6}$$

Thus the potential energy is

$$\mathrm{PE} = \text{force} \times \text{distance} = \frac{1}{2}T\Delta x\left(\frac{\partial A(x,t)}{\partial x}\right)^2 \tag{7}$$

And so

$$\frac{\text{PE}}{\text{length}} = \frac{1}{2} T \left(\frac{\partial A(x,t)}{\partial x} \right)^2 \tag{8}$$

Note that this is proportional to the first derivative squared, not the second derivative. Indeed, even if there were no net force on the test mass at position x (so that $\frac{\partial^2 A}{\partial x^2} = 0$), there would still be potential energy stored in the stretched string. Even though the wave is transverse, the energy comes from stretching the string both longitudinally *and* transversely.

So the total energy per unit length is

$$\frac{E_{\text{tot}}}{\text{length}} = \frac{1}{2} \mu \left(\frac{\partial A}{\partial t} \right)^2 + \frac{1}{2} T \left(\frac{\partial A}{\partial x} \right)^2 \tag{9}$$

Now consider the special case of a traveling wave $A(x,t) = f(x \pm vt)$. Then,

$$\frac{\partial A}{\partial t} = \pm v \frac{\partial A}{\partial x} \tag{10}$$

Thus

$$\left(\frac{\partial A}{\partial x} \right)^2 = \frac{1}{v^2} \left(\frac{\partial A}{\partial t} \right)^2 = \frac{\mu}{T} \left(\frac{\partial A}{\partial t} \right)^2 \tag{11}$$

So the total energy per unit length for a traveling wave

$$\frac{E_{\text{tot}}}{\text{length}} = \frac{1}{2} \mu \left(\frac{\partial A}{\partial t} \right)^2 + \frac{1}{2} T \left(\frac{\partial A}{\partial x} \right)^2 = \mu \left(\frac{\partial A}{\partial t} \right)^2 \tag{12}$$

Recalling that impedance for a string is

$$Z = \frac{\text{force}}{\text{velocity}} = \frac{T}{v} = \sqrt{T\mu} = v\mu \tag{13}$$

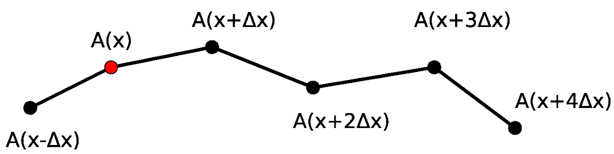
we can write the energy as

$$\frac{E_{\text{tot}}}{\text{length}} = \frac{Z}{v} \left(\frac{\partial A}{\partial t} \right)^2 \tag{14}$$

2 Power

An extremely important quantity related to waves is power. We want to use waves to do things, such as transmit sound or light, or energy in a wire. Thus we want to know the rate at which work can be done using a wave. For example, if you have an incoming sound wave, how much power can be transmitted by the wave to a microphone?

For an incoming traveling wave, let us return to this figure



We want to know how much power can be transmitted from the test mass at $A(x - \Delta x)$ to the test mass at $A(x)$. Now, power = force × velocity. But we don't want the net force on $A(x)$, only the force from the left to compute power transmitted.

We calculated that the force from the left is $F = T\frac{\partial A}{\partial x}$. For $\frac{\partial A}{\partial x} > 0$ this force pulls downward. To see the power transmitted, we need the force which moves the string away from equilibrium, which is the upward force $F = -T\frac{\partial A}{\partial x}$. Then

$$P = F \cdot v = -T \left(\frac{\partial A(x,t)}{\partial x} \right) \left(\frac{\partial A(x,t)}{\partial t} \right) \tag{15}$$

For a traveling wave $A(x,t) = f(x \pm vt)$ we can use Eq. (10) to write this as

$$P = \mp vT\left(\frac{\partial A(x,t)}{\partial x}\right)^2 = \mp v\mu\left(\frac{\partial A(x,t)}{\partial t}\right)^2 = \mp Z\left(\frac{\partial A(x,t)}{\partial t}\right)^2 \tag{16}$$

The sign is + for a right-moving wave (power goes to the right) and − for a left-moving wave.

Now recall that if we have a wave going from a medium with impedance Z_1 into a medium with impedance Z_2, the amplitude of the transmitted and reflective waves are related to the amplitude of the incoming wave by

$$A_T = \frac{2Z_1}{Z_1 + Z_2}A_I, \quad A_R = \frac{Z_1 - Z_2}{Z_1 + Z_2}A_I, \tag{17}$$

Thus if the power in the incoming wave is

$$P_I = Z_1\left(\frac{\partial A_I}{\partial t}\right)^2 \tag{18}$$

Then the power in the reflected wave is

$$P_R = Z_1\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\frac{\partial A_I}{\partial t}\right)^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 P_I \tag{19}$$

and the power in the transmitted wave is

$$P_T = Z_2\left(\frac{2Z_1}{Z_1 + Z_2}\frac{\partial A_I}{\partial t}\right)^2 = \frac{Z_2}{Z_1}\left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 P_I \tag{20}$$

Thus, the fraction of power reflected is

$$\frac{P_R}{P_I} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 = \frac{Z_1^2 - 2Z_1Z_2 + Z_2^2}{(Z_1 + Z_2)^2} \tag{21}$$

and the fraction of power transmitted is

$$\frac{P_T}{P_I} = \frac{Z_2}{Z_1}\left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2} \tag{22}$$

Note that

$$\frac{P_T + P_R}{P_I} = 1 \tag{23}$$

so that, overall, power is conserved.

3 Sound intensity (decibels)

Intensity is defined as power per unit area: $I = \frac{P}{A}$. Sound intensity is measured in **decibels**. A logarithmic scale is used for sound intensity because human hearing is logarithmic. For example, if something has an intensity 1000 times larger, you will perceive it as being 3 times as loud.

Decibel is a logarithmic scale normalized so that 0 dB is $10^{-12}\,\frac{W}{m}$. That is, by definition

$$0\,\mathrm{dB} \equiv 10^{-12}\frac{\mathrm{Watts}}{\mathrm{meter}^2} = I_0 \tag{24}$$

A decibel is 1 tenth of a bel. The number of bels can be computed from a given intensity by

$$\text{loudness in bels} = 100\log_{10}\frac{I}{I_0} \tag{25}$$

The number of decibels is therefore

$$\text{loudness in decibels} = 10\log_{10}\frac{I}{I_0} \tag{26}$$

Some reference intensities are

sound	decibels
Threshold for human hearing	0
Breathing at 3 meters	10
Rustling of leaves	20
...	
Music at 1 meter	70
vacuum cleaner	80
...	
rock concert	120
threshold for pain	130
jet engine at 30 meters	150

Table 1. Reference decibel intensities

It is easy to compute the decibel intensity. For example, suppose you are 3 meters away from a 50 Watt speaker. Watts are a unit of power, so at 3 meters, the power is distributed across a sphere of surface area $A = 4\pi r^2$. Thus, if all the power went into sound, the intensity would be

$$I = \frac{50W}{4\pi(3m)^2} = 0.44\frac{W}{m^2} \tag{27}$$

Thus,

$$L = 10\log_{10}\frac{0.44\frac{W}{m^2}}{10^{-12}\frac{W}{m^2}} = 116\,\mathrm{dB} \tag{28}$$

This is not actually how loud a speaker is (it's more like the loudness of a rock concert). In reality, the energy of the speaker is only transmitted into sound very inefficiently. We can define the efficiency as the power coming out in sound divided by the power which the speaker draws from the battery. The efficiency is around 10^{-5} for a typical speaker. So,

$$L = 10\log_{10}\frac{10^{-5}0.44\frac{W}{m^2}}{10^{-12}\frac{W}{m^2}} = 66\,\mathrm{dB} \tag{29}$$

The efficiency is so low because the speaker and the air have very different impedances.

It takes about 150 mW of power to bow a violin and about 6mW of power comes out in sound. So a violin has an efficiency of $\varepsilon = 0.04 = 4\%$. This is much greater than a speaker, but still most of the energy used in bowing a violin is mechanical and not transmitted into sound.

How does the loudness change with distance? Since $I = \frac{P}{4\pi r^2}$ we have

$$L = 10\log_{10}\frac{P}{4\pi r^2 I_0} = 10\log_{10}\frac{P}{4\pi I_0} - 20\log_{10} r \tag{30}$$

Thus loudness only drops logarithmically with distance. Say you measure a loudness L_0 at a distance r_0. Then the loudness at a distance $2r_0$ would be

$$L = 10\log_{10}\frac{P}{4\pi I_0} - 20\log_{10}(2r_0) = L_0 - 20\log_{10}2 = L_0 - 6.021 \tag{31}$$

That is, it's 6 decibels lower.

Or we can ask at what distance compared to r_0 we should have to go for the loudness to drop by 10 decibels? That would be

$$20\log_{10}\frac{r}{r_0} = 10 \quad \Rightarrow \quad r = 3.162\,r_0 \tag{32}$$

So if you go 3 times farther away, you are down by 10 decibels.

4 Plane waves

The waves will propagate in 3 dimensions, so we need the 3-dimensional version of the wave equation:

$$\left[\frac{\partial^2}{\partial t^2} - v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right]A(x,y,z,t) = 0 \tag{33}$$

This is the obvious generalization of the 1D wave equation. It is invariant under rotations of x, y and z (in fact, it is invariant under a larger group of symmetries, Lorentz transformations, which mix space and time).

There are many solutions to this 3D wave equation. Important solutions are **plane waves**

$$A(x,y,z,t) = A_0\cos\left(\vec{k}\cdot\vec{x} - \omega t + \phi\right) \tag{34}$$

for some amplitude A_0, frequency ω and fixed vector $\vec{k}$ called the **wavevector**. For a plane wave to satisfy the wave equation, its frequency and wavevector must be related by

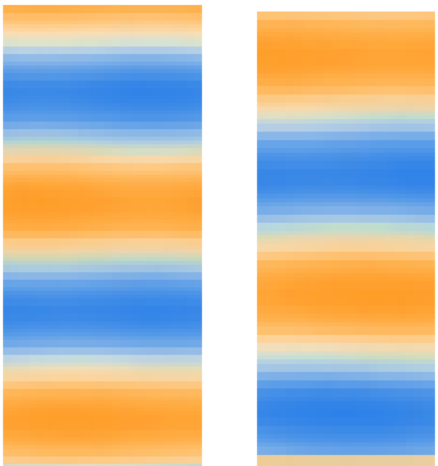
$$\omega = v\left|\vec{k}\right| \tag{35}$$

Thus norm of $\vec{k}$ is fixed by ω and v, but $\vec{k}$ can point in any direction. The direction $\vec{k}$ points is the direction the plane wave is traveling. For example, if $\vec{k} = (0, k, 0)$ the wave is

$$A(x,y,z,t) = A_0\cos(k\,(y - vt) + \phi) \tag{36}$$

which is a wave traveling in the y direction with angular frequency $\omega = kv$. This plane wave is constant in the x and z directions.

Planes waves have **direction** and **phase**. Falstad's ripple simulation (see link on isite) program gives a nice way to visualize waves. It shows the amplitude of the wave by different by colors. Plane waves going in the y direction look like

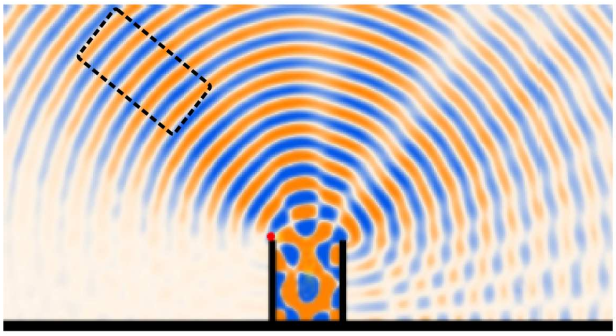


$\cos(ky - \omega t)$ $\cos(ky - \omega t + \pi)$ (37)

These are both plane waves, but they have different phases. These happen to be exactly out of phase.

Plane waves form a basis of all possible solutions to the wave equation. They are the normal modes of the 3D wave equation. For each frequency ω there are plane waves in any direction $\vec{k}$ with $\left|\vec{k}\right| = \frac{\omega}{v}$ with any possible phase.

Another important feature of plane waves is that if you are far enough away from sources, *everything reduces to a plane wave*. For example, if we have some messy source in a cavity, the solution to the wave equation might look like



Inside the dashed box, the solution is very similar to the solution of the plane wave.

How much power is in a plane wave? At a time t the part of the wave in Eq. (36) at position y has power

$$P(t,y) = Z\left(\frac{\partial A}{\partial t}\right)^2 = Z A_0^2 \omega^2 \sin^2(ky - \omega t + \phi) \tag{38}$$

This power is always positive but it oscillates from 0 to its maximum value as the wave oscillates. We don't care so much about these fluctuations. A more useful quantity is the average power. Averaging the power over a wavelength $\lambda = \frac{2\pi}{k}$ gives

$$\langle P \rangle = P_{\text{avg}} = \frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} dy P(t,y) = \frac{1}{2} Z \omega^2 A_0^2 \tag{39}$$

where $Z = \rho c_s$ is the impedance for air. The average power is in principle a function of time. In the case of a plane wave, the average power is time-independent.

5 Interference

Now we are ready to discuss one of the most important concepts in waves (and perhaps all of physics) constructive and destructive interference..

Suppose we have a speaker emitting sound a frequency ω. If the speaker is at $y = 0$ it and we are at large enough distances, the sound will appear as a plane wave

$$A_1 = A_0 \cos(\omega t - ky + \phi_1) \tag{40}$$

Now say we have another speaker directly behind the first speaker producing the same sound at the same volume. We also find a plane wave solution are at large enough distances, the sound will appear as a plane wave

$$A_2 = A_0 \cos(\omega t - ky + \phi_2) \tag{41}$$

We know the frequencies are the same, and k is the same, but the phases can be different. The total wave is then

$$A_{\text{tot}} = A_1 + A_2 = A_0 \cos(\omega t - ky + \phi_1) + A_0 \cos(\omega t - ky + \phi_2) \tag{42}$$

$$= 2A_0 \cos\left(\omega t - ky + \frac{\phi_1 + \phi_2}{2}\right) \cos\left(\frac{\Delta\phi}{2}\right) \tag{43}$$

where $\Delta\phi = \phi_1 - \phi_2$ is the phase difference. This last step is just some trigonometry, which you can check by applying the Mathematica command TrigReduce[] to Eq. (43).

Thus the average power is now

$$\langle P_2 \rangle = \frac{1}{2} Z\omega^2 (2A_0)^2 \cos^2\left(\frac{\Delta\phi}{2}\right) = 4\langle P_1 \rangle \cos^2\left(\frac{\Delta\phi}{2}\right) \tag{44}$$

where $\langle P_1 \rangle = \frac{1}{2} Z\omega^2 A_0^2$ is the average power from a single source.

In a generic situation, say where multiple frequencies are produced by different uncorrelated speakers, the phases will have nothing to do with each other. Then, over time the phase difference can change and we should average the phase difference too. Replacing $\cos^2\left(\frac{\Delta\phi}{2}\right)$ by $\frac{1}{2}$ then gives

$$\langle P_2 \rangle = 4\langle P_1 \rangle \frac{1}{2} = 2\langle P_1 \rangle \tag{45}$$

So two speakers produce twice the power of one speaker. This makes perfect sense.

Now suppose instead that the two speakers are exactly out of phase, so that $\Delta\phi = \pi$, we find

$$\langle P_2 \rangle = 0 \tag{46}$$

Thus no power is emitted. This is **destructive interference**. Conversely, if $\Delta\phi = 0$, then

$$\langle P_2 \rangle = 4\langle P_1 \rangle \tag{47}$$

This is **constructive interference**. Thus two speakers working in phase can produce *four times the power* of a single speaker.

How is this possible? Where is the power coming from? If one looks at how much power is being drawn from the currents driving the speakers, will you find twice as much power is being drawn when the speakers are coherent as when they are incoherent? What is actually happening is that one speaker is pushing down on the other speaker, forcing it to work harder. This is called **source loading**. Thus (in principle) more power is being used by the speakers. However, you won't see this by looking at the power being drawn, since speakers are very inefficient – only 0.01% of the power goes to sound. Instead, the source loading is actually making the speaker more efficient, so that more sound comes out with the same power.

One way to get a produce two coherent speakers is having one speaker a distance d from a wall:

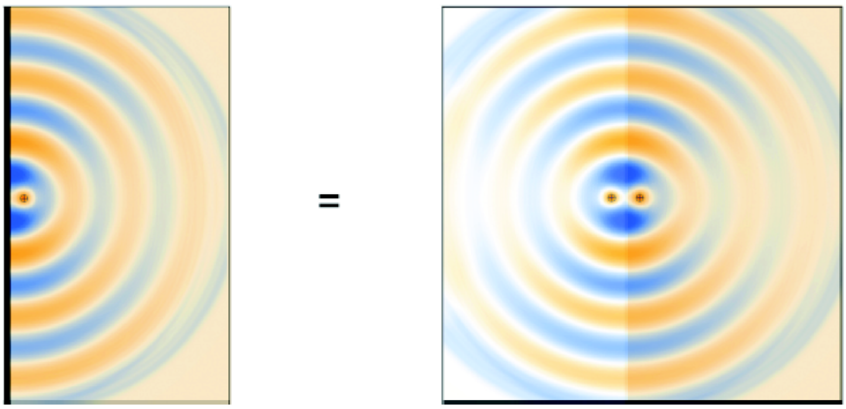


Figure 1. A source near a wall looks like two sources

A wall has infinite impedance ($Z_2 = \infty$), so the sound will reflect off completely. Recall that the amplitude for a sound wave $A(x,t)$ describes the longitudinal displacement at time t of molecules whose equilibrium position is at x. If $A > 0$ is a displacement to the right, then when the wave reflects it will have $A < 0$, since $R = -1$. Thus the reflected wave will be displaced to the left. Thus its displacement looks like the mirror image of what the displacement would be if there were no wall. Thus, at a point x there will be sound coming directly from the source and also sound from the reflection. Say the angular frequency is ω and the wavelength is $\lambda = \frac{2\pi}{k} = 2\pi\frac{v}{\omega}$. What is the intensity picked up by a microphone a distance L from the wall?

The easiest way to compute the effect of the reflection is using the method of images: the reflection will act exactly like a source a distance d on the other side of the wall. Far enough from the source and the wall both the source and its image will produce plane waves. The original source is a distance d from the wall and so a distance $L-d$ from the microphone it produces

$$A_1 = A_0\cos(\omega t - k\,(L-d)) \tag{48}$$

where we have set the $\phi = 0$ for the source for simplicity. Then the image source will produce

$$A_2 = A_0\cos(\omega t - k\,(L+d)) \tag{49}$$

Thus the phase difference is

$$\Delta\phi = 2\,k\,d = 4\pi\frac{d}{\lambda} \tag{50}$$

For $d \ll \lambda$ (the speaker is close to the wall), then $\Delta\phi \approx 0$ and we have complete constructive interference. Thus by putting a speaker near a wall we get **four times the power**. This is called a **proximity resonance** or **self-amplification**.

You might have expected there to be twice the power. Since all the power is going into half the space, by conservation of energy this is perfectly logical. Indeed, if the source and the image were incoherent, there would be twice the power. However, we get another factor of 2 from source loading so in fact the power goes up by 4.

It is natural to try to add more proximity resonances. For example, if we have 4 walls or a 30% wedge we would get

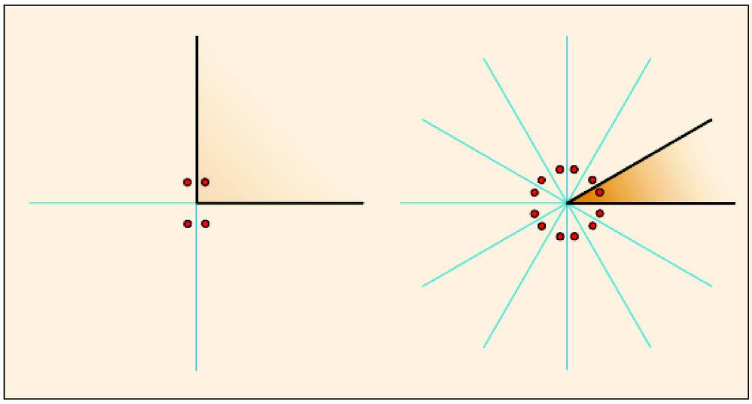


Figure 7.4: Method of images for a single source near a 90° corner and in a 30° wedge. The physical walls are shown as the black lines, and the physical region is shaded. The extra reflection planes for determining the location of the images are shown in light blue. The single source with the walls present or the multiple sources with the walls removed give the same result in the physical region.

This figure and caption are taken from Heller.

With four walls, the area goes to one fourth the area, so naively you would expect a factor of 4 increase in intensity. However, due to source loading the enhancement is a factor of 16. Try standing near a corner and you can hear this yourself. For the 30 degree wedge, source loading gives a factor of 144 enhancement. For a 30 degree wedge in 3 dimensions, the enhancement is around a factor of 200.

The amount of enhancement depends on the frequencies involved. When $d \sim \lambda$ there is as much destructive interference as constructive interference. Thus, there is no source loading or proximity resonances for high frequencies. One still gets the enhancement in intensity from confining the sound to a smaller volume, but this is not as dramatic as when source loading is relevant. When the size of the wedge is of order the distance to the source, boundary effects become important and one cannot use the plane wave approximations. You can play with numerical solutions with arbitrary configurations using Falstad's ripple.