Matthew Schwartz

Lecture 2: Driven oscillators

1 Introduction

We started last time to analyze the equation describing the motion of a damped-driven oscillator:

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2 x=F(t) \tag{1}$$

For small damping $\gamma\ll\omega_0$, we found solutions for $F(t)=0$ of the form

$$x(t)=Ae^{-\frac{\gamma}{2}t}\cos(\omega_0 t+\phi) \tag{2}$$

where the amplitude A and the phase ϕ are determined by initial conditions. Now we will see how to deal with $F(t)$.

We found the damped solution by guessing that an exponential $x(t)=Ae^{\alpha t}$ should work, since its derivatives are all proportional to itself. Plugging this ansatz in with $F(t)$ we find

$$Ae^{\alpha t}(\alpha^2+\gamma\alpha+\omega_0^2)=F(t) \tag{3}$$

This will clearly not be solved for constant α unless $F(t)$ happens to be of the form $e^{\alpha t}$. The trick to solving this equation is to use linearity.

Let us suppose that we can write

$$F(t)=\sum_j c_j\cos(\omega_j t) \tag{4}$$

where c_j are real numbers. It may seem that only a handful of functions can be written this way, but actually *any periodic function* can be written as in Eq. (4) or as

$$F(t)=\sum_j\left[a_j\sin(\omega_j t)+b_j\cos(\omega_j t)\right] \tag{5}$$

with real numbers a_j and b_j. This truly remarkable fact is known as Fourier's theorem, and we will study it soon. For now, let us just take Eq. (4) as given.

Once $F(t)$ is written as a sum of cosines, we can solve the differential equation for each cosine separately then add them. By linearity we then get a solution to the original equation. That is, if we can find functions $x_j(t)$ satisfying

$$\frac{d^2x_j}{dt^2}+\gamma\frac{dx_j}{dt}+\omega_0^2 x_j=\cos(\omega_j t) \tag{6}$$

Multiplying this equation by c_j and summing of j gives

$$\sum_j c_j\left[\frac{d^2x_j}{dt^2}+\gamma\frac{dx_j}{dt}+\omega_0^2 x_j\right]=\sum_j c_j[\cos(\omega_j t)] \tag{7}$$

Therefore, if we define

$$x(t)=\sum c_j x_j(t) \tag{8}$$

we immediately get

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2 x=F(t) \tag{9}$$

as desired.

In summary, assuming Eq. (4) holds (we will come back to this soon), we have reduced solving Eq. (1) to solving Eq. (6). That is, if we can solve the equation with a $\cos(\omega_d t)$ driving force, we can solve the equation for *any* driving force.

2 Driven oscillator

Our first task is to solve

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2\, x = \frac{F_0}{m}\cos(\omega_d t) \tag{10}$$

Here we have made the normalization more physical by adding F_0, for the strength of force with units of force, and dividing by the oscillator mass m to get an acceleration (the left hand side has units of acceleration, as in $\frac{d^2x}{dt^2}$). What's a good guess for a solution? Trying $x(t) = \cos(\omega_d t)$ or $x(t) = \sin(\omega_d t)$ will not work since there are first *and* second derivatives in the equation. We need exponentials.

The key to turning the problem from cosines into exponentials is to recall that

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t) \tag{11}$$

so that

$$\cos(\omega_d t) = \mathrm{Re}(e^{-i\omega_d t}) \tag{12}$$

Now suppose we find a solution to

$$\frac{d^2}{dt^2}z + \gamma\frac{d}{dt}z + \omega_0^2\, z = \frac{F_0}{m}e^{-i\omega_d t} \tag{13}$$

with a complex function $z(t)$. Then we define

$$x(t) \equiv \mathrm{Re}\,[z(t)] \tag{14}$$

Taking the real part of Eq. (13) then gives

$$\mathrm{Re}\left[\frac{d^2}{dt^2}z + \gamma\frac{d}{dt}z + \omega_0^2\, z\right] = \mathrm{Re}\left[\frac{F_0}{m}e^{-i\omega_d t}\right] \tag{15}$$

which is exactly Eq. (10). So we have reduced the problem to using an exponential driving force instead of a cosine driving force.

Plugging in a guess $z(t) = Ce^{-i\omega_d t}$ into Eq. (15) gives

$$Ce^{-i\omega_d t}[-\omega_d^2 - i\gamma\omega_d + \omega_0^2] = \frac{F_0}{m}\, e^{-i\omega_d t} \tag{16}$$

Now the $e^{i\omega_d t}$ factors drop out and we have a simple algebraic relation

$$C = \frac{F_0}{m}\frac{1}{\omega_0^2 - i\gamma\omega_d - \omega_d^2} \tag{17}$$

Thus

$$z(t) = \frac{F_0}{m}\frac{1}{\omega_0^2 - i\gamma\omega_d - \omega_d^2}e^{-i\omega_d t} \tag{18}$$

To get the solution to the original equation with a real function $x(t)$ we use Eq. (14):

$$x(t) = \mathrm{Re}\left[\frac{F_0}{m}\frac{1}{\omega_0^2 - i\gamma\omega_d - \omega_d^2}e^{-i\omega_d t}\right] \tag{19}$$

Now we just have to simplify this using algebra.

First, we get the $i's$ to the numerator by writing

$$\frac{1}{\omega_0^2 - i\gamma\omega_d - \omega_d^2} = \frac{\omega_0^2 - \omega_d^2 + i\gamma\omega_d}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d}\frac{1}{\omega_0^2 - \omega_d^2 - i\gamma\omega_d} = \frac{\omega_0^2 - \omega_d^2 + i\gamma\omega_d}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2} \tag{20}$$

$$\equiv A + Bi \tag{21}$$

where

$$A = \frac{\omega_0^2 - \omega_d^2}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2} \qquad B = \frac{\gamma\omega_d}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2} \tag{22}$$

Then

$$x(t) = \mathrm{Re}\left[\frac{F_0}{m}(A + Bi)e^{-i\omega_d t}\right] = \frac{F_0}{m}\mathrm{Re}[(A + Bi)(\cos(\omega_d t) - i\sin(\omega_d t))] \tag{23}$$

$$= \frac{F_0}{m}(A\cos\omega_d t + B\sin\omega_d t) \tag{24}$$

In summary, we found an exact solution to Eq. (10):

$$\boxed{x(t) = \frac{F_0}{m}\left\{\frac{\omega_0^2 - \omega_d^2}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2}\cos\omega_d t + \frac{\gamma\omega_d}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2}\sin\omega_d t\right\}} \tag{25}$$

2.1 Transients

We found a single exact solution. What happened to the boundary conditions? The dependence on boundary conditions is entirely determined by solutions to the **homogeneous** equation, with $F = 0$:

$$\frac{d^2 x_0}{dt^2} + \gamma\frac{dx_0}{dt} + \omega_0^2 x_0 = 0 \tag{26}$$

Solutions to this equation are called **homogeneous solutions**. The solution $x(t)$ in Eq. (25) is called the **inhomogeneous solution**. Note that $x_0(t) + x(t)$ will also satisfy the inhomogeneous Eq. (10), due to linearity. Thus we can always add a homogeneous solution to an inhomogeneous solution. We saw before that the homogeneous solutions all have $e^{-\frac{\gamma}{2}t}$ factors, plus possibly some oscillatory component. Thus they die off at late time. For this reason, they are called **transient**. Transients are determined by boundary conditions. If you have a driving force for long enough time, then the transient is irrelevant.

2.2 Phase lag

A good way to see the physics hidden in the solution $x(t)$ is to take limits. First, consider the limit with no damping, $\gamma = 0$. Then,

$$x(t) = \frac{F_0}{m}\frac{1}{\omega_0^2 - \omega_d^2}\cos\omega_d t \tag{27}$$

We can compare this to our driving force $F(t) = F_0\cos\omega_d t$. For $\omega_d < \omega_0$ the sign of the position and the force are the same so they are exactly in phase. Now say we crank up the driving frequency ω_d until it reaches then surpasses ω_0. For $\omega_d > \omega_0$, the sign of the solution flips and the oscillator is out out of phase with the driver. Physically, the oscillator can't keep up with the driving force: it experiences **phase lag**.

2.3 Power and energy

We see from Eq. (25) there is a part of $x(t)$ which is exactly proportional to the driving force $F(t) = F_0\cos\omega_d t$ and a part which is out of phase. We call the in-phase part the **elastic amplitude**. It is proportional to

$$A = \frac{\omega_0^2 - \omega_d^2}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2} \tag{28}$$

The out-of-phase part is the **absorptive amplitude**. Its magnitude is

$$B = \frac{\gamma\omega_d}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2} \tag{29}$$

Thus for $\gamma = 0$, no damping, there is no absorptive part. Since the absorptive part is proportional to γ it should have to do with energy being lost from the oscillator into the system. To see how this works, we need to compute the energy and the power.

Recall that work is force times displacement $W = F\Delta x$ and power is work per unit time:

$$P = \frac{W}{\Delta t} = F\frac{\Delta x}{\Delta t} \tag{30}$$

For small displacements and small times, this becomes

$$P = F\frac{dx}{dt} \tag{31}$$

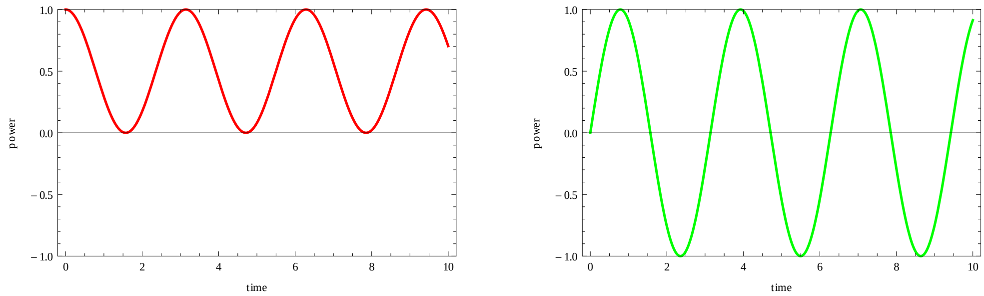
Plugging in our solution $x(t) = \frac{F_0}{m}[A\cos(\omega_d t) + B\sin(\omega_d t)]$

$$P = F_0\cos(\omega_d t)\left[-\omega_d \frac{F_0}{m} A\sin(\omega_d t) + \omega_d \frac{F_0}{m} B\cos(\omega_d t)\right] \tag{32}$$

$$= -\frac{F_0^2}{2m}\omega_d A\sin(2\omega_d t) + \frac{F_0^2}{m} B\omega_d \cos^2(\omega_d t) \tag{33}$$

where $2\sin\theta\cos\theta = \sin(2\theta)$ has been used. This is the **power put into the system by the driving force**.

We see that the absorptive part is proportional to $\cos^2\omega_d t$ which is positive for all times. Thus it always takes (absorbs) power. On the other hand, the elastic amplitude is proportional to $\sin(2\omega_d t)$ which is sometimes positive and sometimes negative. These are shown here



absorptive amplitude
always takes power (P>0)

elastic amplitude
sometimes takes power (P>0)
sometimes gives power back (P<0)

Figure 1. Absorptive and elastic amplitudes

When the power is negative, as in the elastic amplitude, the oscillator is returning power to the driver. The elastic amplitude averages to zero. Since $\gamma = 0$ implies that the absorptive amplitude vanishes so the entire solution is elastic, we draw the logical conclusion that with no damping ($\gamma = 0$) no net power is needed to drive the system (a little power is needed to get it started, but once it's moving, the driver no longer does work).

The average power put into the system is over a period $T = \frac{2\pi}{\omega_d}$ is

$$\langle P \rangle = \frac{1}{T}\int_0^T dt\, P(t) = \frac{F_0^2}{2m} B\omega_d = \left(\frac{F_0^2}{2\gamma m}\right)\frac{(\gamma\omega_d)^2}{(\omega_0^2 - \omega_d^2)^2 + (\gamma\omega_d)^2} \tag{34}$$

Here is a plot of this average power as a function of ω_d for fixed γ and ω_0.

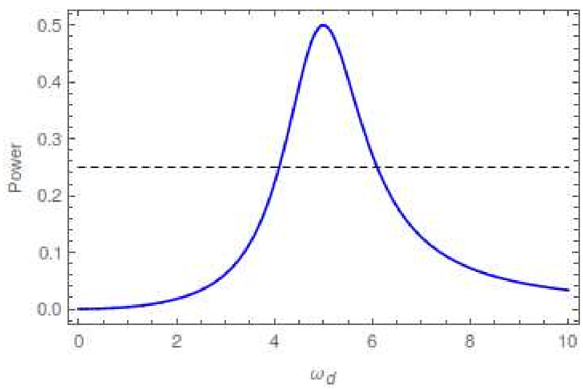


Figure 2. Power absorbed for $\gamma = 2$ and $\omega_0 = 5$ as a function of the driving frequency ω_d. The maximum is when $\omega_d = \omega_0$ known as **resonance**. The dashed line is half of the maximum power. The length of the dashed line between the points where it hits the curve (width at half-maximum) is γ.

This power absorption curve has a maximum at $\omega_d = \omega_0$ (you can check this) where $\langle P \rangle = \frac{F_0^2}{2\gamma m}$. This is known as **a resonance**.γ. One way to find the resonance frequency ω_0 of a system is by varying the driving force until maximum power is absorbed. The power is half the resonant power, $\langle P \rangle = \frac{F_0^2}{4\gamma m}$, when

$$\omega_d = \frac{1}{2}\sqrt{4\omega_0^2 + \gamma^2} \pm \frac{1}{2}\gamma \tag{35}$$

The difference between these two driving frequencies is γ. Thus, one can also read γ off of the plot in Fig. 2: it is the value of the width at half-maximum. This kind of curve is called a **Lorentzian**. Its maximum is at ω_0 and its width is γ.