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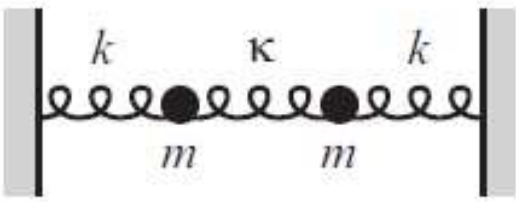
Lecture 3: Coupled oscillators

1 Two masses

To get to waves from oscillators, we have to start coupling them together. In the limit of a large number of coupled oscillators, we will find solutions while look like waves. Certain features of waves, such as resonance and normal modes, can be understood with a finite number of oscillators. Thus we start with two oscillators.

Consider two masses attached with springs



$$ \tag{1}$$

Let's say the masses are identical, but the spring constants are different.

Let x_1 be the displacement of the first mass from its equilibrium and x_2 be the displacement of the second mass from its equilibrium. To work out Newton's laws, we first want to know the force on x_1 when it is moved from its equilibrium while holding x_2 fixed. This is

$$F_{\text{on 1 from moving 1}} = F = -kx_1 - \kappa x_1 \tag{2}$$

The signs are both chosen so that they oppose the motion of the mass. There is also a force on x_1 if we move x_2 holding x_1 fixed. This force is

$$F_{\text{on 1 from moving 2}} = \kappa x_2 \tag{3}$$

To check the sign, note that if x_2 is increased, it pulls x_1 to the right. There is no contribution to this force from the spring between the second mass and the wall, since we are moving the mass by hand and just asking how it affects the first mass. Thus

$$m\,\ddot{x}_1 = -(k+\kappa)x_1 + \kappa x_2 \tag{4}$$

similarly,

$$m\,\ddot{x}_2 = -(k+\kappa)x_2 + \kappa x_1 \tag{5}$$

One way to solve these equations is to note that if we add them, we get

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \tag{6}$$

This is just $m\,\ddot{y} = -ky$ for $y = x_1 + x_2$, so the solutions are sines and cosines, or cosine and a phase:

$$x_1 + x_2 = A_s \cos(\omega_s t + \phi_s), \quad \omega_s = \sqrt{\frac{k}{m}} \tag{7}$$

Another way solve them is taking the difference

$$m(\ddot{x}_1 - \ddot{x}_2) = (-k - 2\kappa)(x_1 - x_2) \quad \Rightarrow \quad x_1 - x_2 = A_f \cos(\omega_f t + \phi_f), \quad \omega_f = \sqrt{\frac{k+2\kappa}{m}} \tag{8}$$

We write ω_s for ω_{slow} and ω_f for ω_{fast}, since $\omega_f > \omega_s$. Thus we have found two solutions each of which oscillate with fixed frequency. These are the **normal modes** for this system. A general solution is a linear combination of these two solutions. Explicitly, we have:

$$x_1 = \frac{1}{2}[(x_1 + x_2) + (x_1 - x_2)] = \frac{1}{2}[A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f)] \tag{9}$$

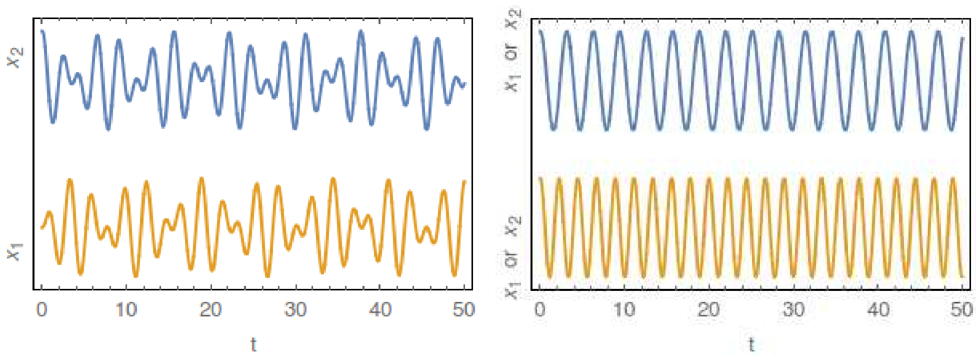
$$x_2 = \frac{1}{2}[(x_1 + x_2) - (x_1 - x_2)] = \frac{1}{2}[A_s \cos(\omega_s t + \phi_s) - A_f \cos(\omega_f t + \phi_f)] \tag{10}$$

If we can excite the masses so that $A_f = 0$ then the masses will both oscillate at the frequency ω_s. In practice, we can do this by pulling the masses to the right by the same amount, so that $x_1(0) = x_2(0)$ which implies $A_f = 0$. The solution is then $x_1 = x_2$ and both oscillate at the frequency A_s for all time. This is the **symmetric oscillation mode.** Since $x_1 = x_2$ at all times, both masses move right together, then move left together.

If we excite the masses in such a way that $A_s = 0$ then $x_1 = -x_2$ and both oscillate at frequency ω_f. We can set this up by pulling the masses in opposite directions. In this mode, when one mass is right of equilibrium, the other is left, and vice versa. So this is an **antisymmetric mode**.

2 Beats

You should try playing with the coupled oscillator solutions in the Mathematica notebook oscillators.nb. Try varying κ and k to see how the solution changes. For example, say $m = 1$, $\kappa = 2$ and $k = 4$. Then $\omega_s = 2$ and $\omega_f = 2\sqrt{2}$, Here are the solutions:

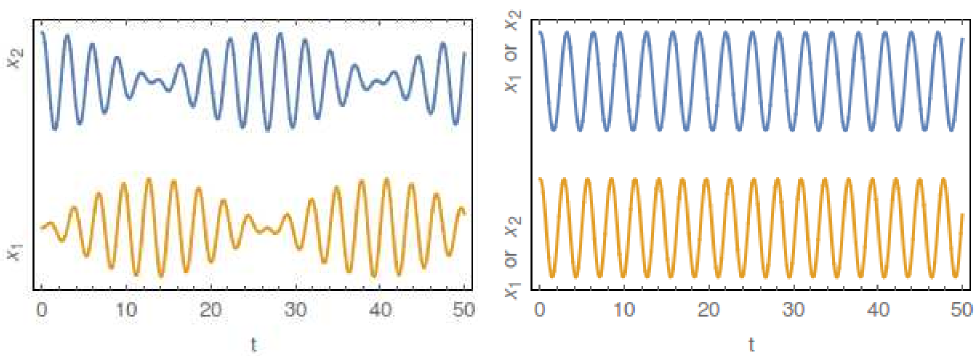


Behavior starting from $x_1 = 1, x_0 = 0$ Normal mode behavior

Figure 1. Left shows the motion of masses $m = 1, \kappa = 2$ and $k = 4$ starting with $x_1 = 1$ and $x_2 = 0$. Right shows the normal modes, with $x_1 = x_2 = 1$ (top) and $x_1 = 1, x_2 = -1$ (bottom).

If you look closely at the left plot, you can make out two distinct frequencies: the normal mode frequencies, as shown on the right.

Now take $\kappa = 0.5$ and $k = 4$. Then $\omega_s = 2$ and $\omega_f = 2.2$. In this case



Behavior starting from $x_1 = 1, x_0 = 0$ Normal mode behavior

Figure 2. Motion of masses and normal modes for $k = 0.5$ and $\kappa = 4$

Now we can definitely see two distinct frequencies in the positions of the two masses. Are these the two frequencies ω_s and ω_f? Comparing to the normal mode plots, it is clear they are not. One is much slower. However, we do note that $\omega_s \approx \omega_f$. What we are seeing here is the emergence of **beats**. Beats occur when two normal mode frequencies get close.

Beats can be understood from the simple trigonometric relation

$$\cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \tag{11}$$

When you excite two frequencies ω_1 and ω_2 at the same time, the solution to the equations of motion is the sum of the separate oscillating solutions (by linearity!). Eq. (11) shows that this sum can also be written as the *product* of two cosines. In particular, if $\omega_1 \approx \omega_2$ then

$$\omega = \frac{\omega_1 + \omega_2}{2} \approx \omega_1 \approx \omega_2 \quad \varepsilon = \frac{\omega_1 - \omega_2}{2} \ll \omega_1, \omega_2 \tag{12}$$

So the sum looks like an oscillation whose frequency ω is the *average* of the two normal mode frequencies modulated by an oscillation with frequency ε given by half the difference in the frequencies.

Beats are important because they can generate frequencies well below the normal mode frequencies. For example, suppose you have two strings which are not quite in tune. Say they are supposed to both be the note $A4$ at 440 Hz, but one is actually $\nu_1 = 442\,\text{Hz}$ and the other is $\nu_2 = 339\,\text{Hz}$. If you pluck both strings together you will hear the average frequency $\Omega = 440.5\,\text{Hz}$, but also there will be an oscillation at $\varepsilon = \frac{1}{2}(442 - 339)\text{Hz} = 1.5\,\text{Hz}$. This oscillation is the enveloping curve over the high frequency (440.5 Hz) oscillations

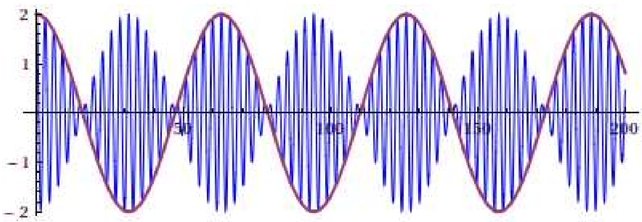


Figure 3. The red curve is $\cos\left(2\pi \frac{\nu_1 - \nu_2}{2} t\right)$. When hearing beats, the observed frequency is the frequency of the extrema $\nu_{\text{beat}} = \nu_1 - \nu_2$ which is twice the frequency of this curve .

As you can see from the figure, due to the high frequency oscillations, there are peaks in the amplitude twice as often as peaks in $\cos\left(2\pi \frac{\nu_1 - \nu_2}{2} t\right)$. Thus what we hear are beats at the **beat frequency**

$$\nu_{\text{beat}} = |\nu_1 - \nu_2| \tag{13}$$

We use an absolute value since we want a frequency to be positive (it's the same frequency whether $\nu_1 > \nu_2$ or $\nu_2 > \nu_1$). Note that there is no factor of 2 in the conventional definition of ν_{beat}, since we only ever hear the modulus of the oscillation not the phase.

Thus with $\nu_f = 442\,\text{Hz}$ and $\nu_s = 339\,\text{Hz}$ the beat frequency is $\nu_{\text{beat}} = 3\,\text{Hz}$. Thus you hear something happening 3 times a second. This is a regular beating in off-tune notes which is audible by ear. In fact, it is a useful trick for tuning – change one string until the beating disappears. Then the strings are in tune. We will see numerous examples of beats as the course progresses.

3 Two masses with matrices

We solved the two coupled mass problem by looking at the equations and noting that their sum and difference would be independent solutions. For more complicated systems (more masses, different couplings) we should not expect to be able to guess the answer in this way. Can you guess the solution if the two oscillators have different masses?

To develop a more systematic procedure, suppose we have lots of masses with lots of different springs connected in a complicated way. Then the equations of motion are

$$m_1\ddot{x}_1 = k_{11}\, x_1 + k_{12}\, x_2 + \cdots + k_{1n} x_n \tag{14}$$

$$\cdots \tag{15}$$

$$m_n\ddot{x}_n = k_{n1}\, x_1 + k_{n2}\, x_2 + \cdots + k_{\mathrm{nn}} x_n \tag{16}$$

where k_{ij} are constants, representing the strength of the spring between masses i and j. Note that all of these equations are linear. What are the solutions in this general case? This is an algebra problem involving linear equations. Hence we should be able to solve it with **linear algebra**.

To connect to linear algebra, let's return to our two mass system. Since the equations of motion are linear, we expect them to be solved by exponentials $x_1 = c_1 e^{i\omega t}$ and $x_2 = c_2 e^{i\omega t}$ for some ω, c_1 and c_2. As with the driven oscillator from the last lecture, we are using complex solutions to make the math simpler, then we can always take the real part at the end. Plugging in these guesses, Eqs. (4) and (5) become

$$-m_1\omega^2 c_1 = -(k+\kappa)c_1 + \kappa c_2 \tag{17}$$

$$-m_2\omega^2 c_2 = -(k+\kappa)c_2 + \kappa c_1 \tag{18}$$

We have let the masses be different for generality.

Next, we will write these equations in matrix form. To do so, we define a vector $\vec{c}$ as

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{19}$$

Then the equations of motion become

$$M \cdot \vec{c} = \begin{pmatrix} \frac{-k-\kappa}{m_1} & \frac{\kappa}{m_1} \\ \frac{\kappa}{m_2} & \frac{-k-\kappa}{m_2} \end{pmatrix} \cdot \vec{c} = -\omega^2 \vec{c} \tag{20}$$

where M is defined by this equation.

You might recognize this as an eigenvalue equation. An $n \times n$ matrix A has n **eigenvalues** λ_i and n associated **eigenvectors** $\vec{v}_i$ which satisfy

$$A \cdot \vec{v}_i = \lambda_i \vec{v}_i \tag{21}$$

The eigenvalues don't all have to be different. Note that the left hand side is a matrix multiplying a vector while the right-hand side is just a number multiplying a vector. So studying eigenvalues and eigenvectors lets us turn matrices into numbers! Eigenvalues and eigenvectors are *the* fundamental mathematical concept of quantum mechanics. I cannot emphasize enough how important it is to master them.

Let's recall how to solve an eigenvalue equation. The trick is to write it first as

$$(A - \lambda \mathbb{1})\vec{v} = 0 \tag{22}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. For $n = 2$, $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For most values of λ, the matrix $(A - \lambda \mathbb{1})$ has an inverse. Multiplying both sides of Eq. (22) by that inverse, we find $\vec{v} = 0$. This is the trivial solution (it obviously satisfies Eq. (21) for any A). The nontrivial solutions consequently must correspond to values of λ for which $(A - \lambda \mathbb{1})$does **not** have an inverse. When does a matrix not have an inverse? A result from linear algebra is that a matrix is not invertible if and only if its determinant is zero. Thus the equation $\det(A - \lambda \mathbb{1}) = 0$ is an algebraic equation for λ whose solutions are the eigenvalues λ_i.

It is useful to know that determinant of a 2×2 matrix is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{23}$$

You should have this memorized. For a 3×3 matrix, the determinant is:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei - fh) - b(di - fg) + c\,(dh - eg) \tag{24}$$

You should know how to compute this, but don't need to memorize the formula. Beyond 3×3, you probably want to take determinants with Mathematica rather than by hand.

So, returning to Eq. (20), the eigenvalues $-\omega^2$ must satisfy

$$0 = \det(M + \omega^2\mathbb{1}) = \det\begin{pmatrix} \frac{-k-\kappa}{m_1} + \omega^2 & \frac{\kappa}{m_1} \\ \frac{\kappa}{m_2} & \frac{-k-\kappa}{m_2} + \omega^2 \end{pmatrix} \tag{25}$$

$$= \left(\frac{-k-\kappa}{m_1} + \omega^2\right)\left(\frac{-k-\kappa}{m_2} + \omega^2\right) - \frac{\kappa^2}{m_1 m_2} \tag{26}$$

This is a quadratic equation for ω^2, with two roots: the two eigenvalues.

Let's set $m_1 = m_2 = m$ now to check that we reproduce our old result. Multiplying Eq. (26) by m^2, it reduces to

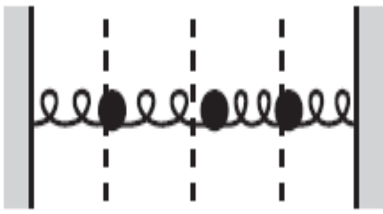
$$(k + \kappa - m\omega^2)^2 = \kappa^2 \tag{27}$$

Thus $k + \kappa - m\omega^2 = \pm\kappa$. Or in other words

$$\omega = \omega_s = \sqrt{\frac{k}{m}}, \quad \omega = \omega_f = \sqrt{\frac{k + 2\kappa}{m}} \tag{28}$$

These are the two normal mode frequencies we found above. Note that we didn't have to take the real part of the solution to find the normal mode frequencies. We only need to take the real part to find the solutions $x(t)$.

Now let's try three masses. We can couple them all together and to the walls in any which way



(29)

The equations of motion for this system will be of the form

$$m_1\ddot{x}_1 = k_{11}\,x_1 + k_{12}\,x_2 + k_{13}x_3 \tag{30}$$

$$m_2\ddot{x}_3 = k_{21}\,x_1 + k_{22}\,x_2 + k_{23}x_3 \tag{31}$$

$$m_3\ddot{x}_3 = k_{31}\,x_1 + k_{32}\,x_2 + k_{33}x_3 \tag{32}$$

Some of these k_{ij} are probably zero, but we don't care. Writing $x_1 = c_1 e^{i\omega t}$, $x_2 = c_2 e^{i\omega t}$ and $x_3 = c_3 e^{i\omega t}$, these equations become algebraic:

$$-\omega^2 c_1 = \frac{k_{11}}{m_1}\,c_1 + \frac{k_{12}}{m_1}c_2 + \frac{k_{13}}{m_1}c_3 \tag{33}$$

$$-\omega^2 c_2 = \frac{k_{21}}{m_2}\,c_1 + \frac{k_{22}}{m_2}\,c_2 + \frac{k_{23}}{m_2}c_3 \tag{34}$$

$$-\omega^2 c_3 = \frac{k_{31}}{m_3}c_1 + \frac{k_{32}}{m_3}c_2 + \frac{k_{33}}{m_3}c_3 \tag{35}$$

In other words,

$$(M + \omega^2\mathbb{1})\vec{x} = 0 \tag{36}$$

with M the matrix whose entries are $M_{ij} = \frac{k_{ij}}{m_i}$. So to find the normal mode frequencies ω, we need to solve $\det(M + \omega^2 \mathbb{1}) = 0$. For a 3×3 matrix, there will be 3 eigenvalues and hence three normal-mode frequencies.