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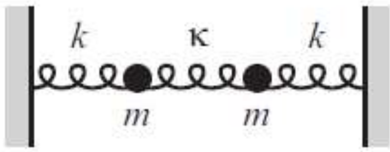
Lecture 4: Oscillators to Waves

1 Review two masses

Last time we studied how two coupled masses on springs move



If we take $\kappa = k$ for simplicity, the two normal modes correspond to

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{Re}[e^{i\omega_s t}], \quad \omega_s = \sqrt{\frac{k}{m}} \tag{1}$$

and

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{Re}[e^{i\omega_f t}], \quad \omega_f = \sqrt{\frac{3k}{m}} \tag{2}$$

One way to make sure we only excite these modes is to displace the masses so that their initial conditions are either $x_1 = x_2$, for the symmetric (slow) mode or $x_1 = -x_2$ if we want to excite only the antisymmetric (fast mode). In other words, the normal mode solutions are in 1-to-1 correspondence with initial conditions. We can draw the initial conditions as

symmetric mode asymmetric mode

Figure 1. Initial conditions to excite normal modes

These pictures are going to be a little hard to look at if there are multiple masses. Thus it is helpful to draw the initial conditions as points in the y direction. That is, we write

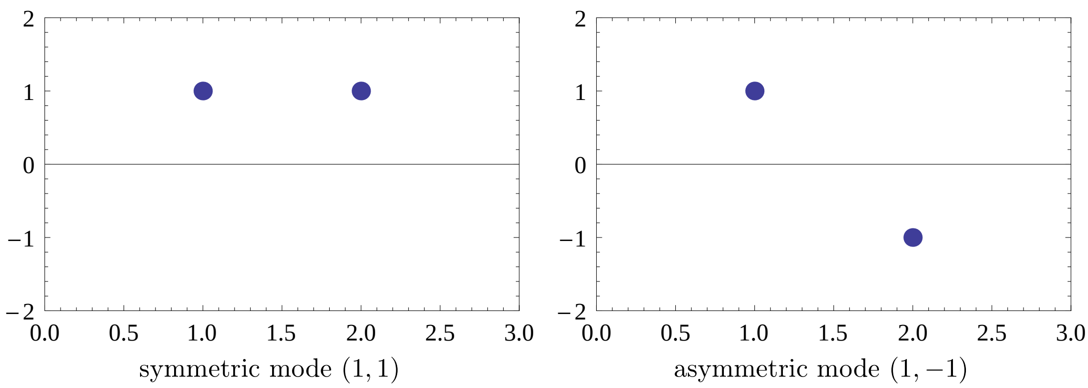
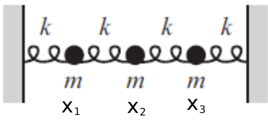


Figure 2. Initial conditions to excite normal modes. The x-axis in these plots is the index of the mass (only two masses, m_1 and m_2 in this case). The y-axis is the displacement from equilibrium that you can release the masses from to get them oscillating with a normal mode frequency. In other words, these are plots of the eigenvectors.

In these figures, the displacement is still **longitudinal** (in the direction of the spring), we are just drawing it in the y axis because it is easier to see.

2 Three masses

Now consider 3 identical masses with all identical spring constants



(3)

The equations of motion for the first mass are

$$m\frac{d^2x_1}{dt^2} = -2kx_1 + kx_2 \tag{4}$$

As before, you should think of first term on the right as the force generated on mass 1 when it is moved by a distance x_1. The sign on $-2kx_1$ is negative since it always wants to go back to equilibrium. The second term $+kx_2$ is the force that is exerted on mass 1 when mass 2 is moved *holding everything else fixed*. It has a + sign, since if I move mass 2, then mass 1 wants to leave its equilibrium position. If we move mass 3 holding everything else fixed, no force is exerted on mass 1.

Similarly,

$$m\frac{d^2x_2}{dt^2} = -2kx_2 + kx_1 + kx_3 \tag{5}$$

and

$$m\frac{d^2x_3}{dt^2} = -2kx_3 + kx_2 \tag{6}$$

Writing

$$\vec{x} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} e^{i\omega t} \tag{7}$$

The equations of motion become

$$-\omega^2 \vec{x} = \omega_0^2 \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} \tag{8}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{9}$$

is the frequency associated with a single mass.

The normal mode frequencies are the eigenvalues of this matrix. Plugging in to Mathematica we find eigenvalues and associated eigenvectors

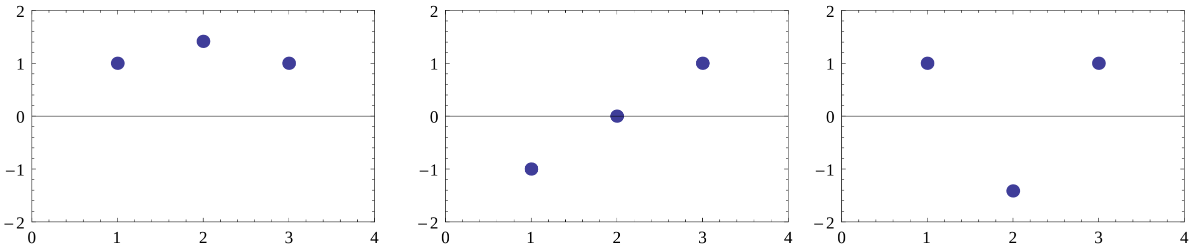
$$\omega_1 = \omega_0\sqrt{2-\sqrt{2}}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \tag{10}$$

$$\omega_2 = \omega_0\sqrt{2}, \quad \vec{x}(0) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{11}$$

$$\omega_3 = \omega_0\sqrt{2+\sqrt{2}}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \tag{12}$$

We have ordered these from slowest to fastest.

To think about these solutions, it is helpful to plot the eigenvectors (the initial displacements)



$\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$ $\qquad \omega_2 = \omega_0\sqrt{2}$ $\qquad \omega_3 = \omega_0\sqrt{2+\sqrt{2}}$

Figure 3. Initial conditions to excite normal modes with 3 masses

From these pictures, it is not hard to see why the frequencies are higher for the third solution - the masses are more stretched apart, so there is more force between them, causing faster oscillation.

3 Completeness of eigenvectors

Before going on to N modes, it's worth making one very important point. Recall from linear algebra that the set of eigenvectors of any matrix is **complete**, meaning that any vector can be written as a linear combination of eigenvectors. For oscillators, this means that any solution to the equations of motion can be written as

$$\vec{x}(t) = \sum_j c_j \vec{x}_j(t) \tag{13}$$

where the sum is over normal modes n. The normal modes are solutions $\vec{x}_j(t)$ to the equation of motion with frequencies ω_j. That is

$$\vec{x}_j(t) = \vec{x}_j^0 \cos(\omega_j t) \tag{14}$$

where $\vec{x}_j^0$ is a constant vector.

In summary, normal modes oscillate with a single frequency. A general solution can always be written as a sum of normal modes.

4 *N* modes

Now we'll solve the N mass system. You should think of lots of springs put together as being simply one long spring where the masses are pieces of the spring itself. We'll see the wave equation result from this system. Solutions to the wave equation describe not just normal modes, but also waves, such as pulses sent down the spring (like pulses sent down a slinky). These pulses are called traveling waves, which are actually linear combinations of normal modes. Understanding waves will occupy the rest of the course, but first we have to solve the N spring system.

We'll construct the equations of motion for the N springs. Then we'll solve the coupled equations numerically for finite N to get a sense for what the answer should look like. Then we'll solve the system exactly for any N.

You may find this section quite abstract. It is not critical that you follow all the details here and be able to reproduce it all on your own. This is one of the most complicated derivations we will do in the course. Do your best. You should understand the result though, as summarized in Section 4.4.

4.1 Equations of motion

Ok, so, we want to string N masses together. Adding more masses to the right of mass 3 does not affect the equations of motions for masses 1 and 2. So their equations are

$$m\frac{d^2x_1}{dt^2} = -2kx_1 + kx_2 \tag{15}$$

and

$$m\frac{d^2x_2}{dt^2} = kx_1 - 2kx_2 + kx_3 \tag{16}$$

as before. Mass 3 is now like mass 2 – it has masses to the right and left of it. Thus,

$$m\frac{d^2x_3}{dt^2} = kx_2 - 2kx_3 + kx_4 \tag{17}$$

In fact, it is easy to see that the generalization for any of the middle masses is

$$\boxed{m\frac{d^2x_n}{dt^2} = kx_{n-1} - 2kx_n + kx_{n+1}} \tag{18}$$

The last mass has no mass on its right, so it gets an equation like mass 1:

$$m\frac{d^2x_N}{dt^2} = kx_{N-1} - 2kx_N \tag{19}$$

Eq. (18), (15) and (19) are what we want to solve.

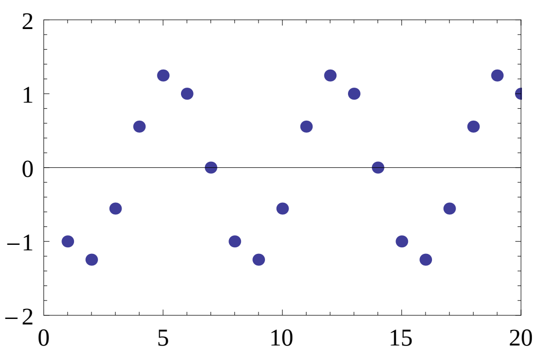
Putting all of these equations together with time dependence $e^{i\omega t}$ for all masses leads to the matrix equation

$$-\omega^2\vec{x} = \omega_0^2 \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \vec{x} \tag{20}$$

with $\omega_0 = \sqrt{\frac{k}{m}}$ as usual. All the entries not shown are zero.

4.2 Numerical solutions

First, let's solve for the eigenvalues and eigenvectors of this system numerically using Mathematica. With 20 masses, the displacements associated with the 15th eigenvalue is



(21)

You may notice that it looks a lot like a cosine curve.

For the 6 mass system, we can plot the displacements for all the normal modes at the same time. Here they are, with the dots for clarity:

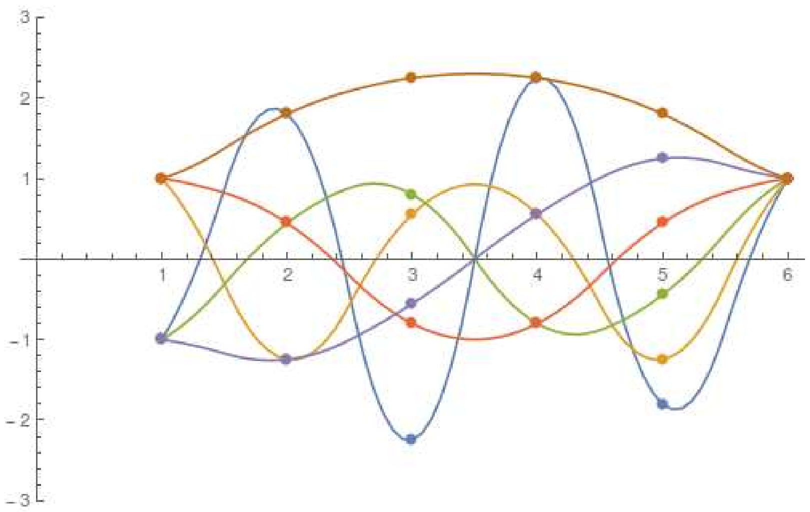


Figure 4. Normal modes displacements for the 6 mass system. These curves look like sine curves.

Already with six masses, in Figure 4, we see that the normal modes look like sine and cosine curves. They are not complete periods though – they stop abruptly. This is due to the equations of motion for masses 1 and N being determined by Eqs. (15) and (19) rather than the equation (18) that the rest of the masses satisfy. What is special about 1 and N is that they are attached to rigid walls, while all the other masses are attached to springs only. These rigid walls correspond to fixed boundary conditions at $n=0$ and $n=7$:

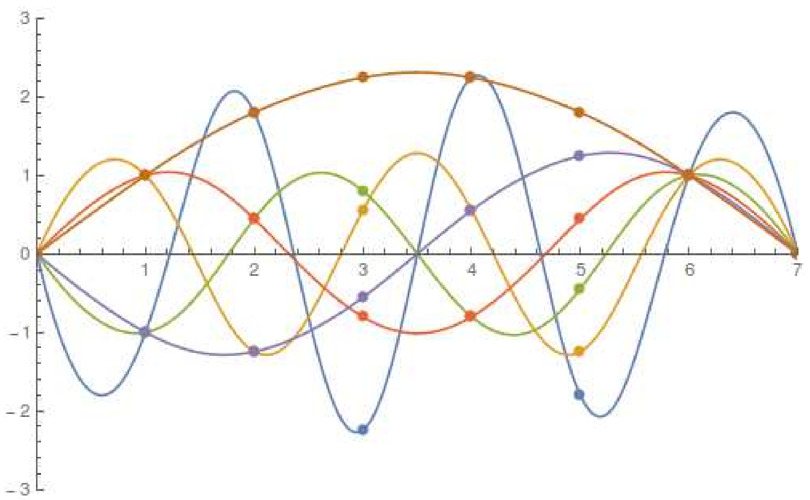


Figure 5. Same as Figure 4, but with interpolations extended to $n=0$ and $n=7$ to show the boundary conditions.

To be clear, there is really no mass at $n=0$, but we are just pretending one is there. Now we can see that the solutions look like

$$x_n = B\sin(pn)e^{i\omega t} \tag{22}$$

for some p. The boundary conditions imply that $p=\frac{\pi}{N+1}j$ for some $j=1,2,3,.....$ These p are called **wavenumbers**. In the continuum limit, we will see that wavenumber $p=\frac{2\pi}{\lambda}$ with λ the wavelength. In the discrete case, p is dimensionless so it's harder to think of it as related to a wavelength. The fact that the wavenumbers are quantized by the boundary conditions is extremely important, both classically and in quantum mechanics. We will be revisiting this quantization at length throughout the course.

4.3 Exact solution

With the numerical solution giving a hint of where to look, let us just solve the system. We want to find vectors x_n that satisfy Eq. (18):

$$\frac{d^2x_n}{dt^2} = \omega_0^2 [x_{n-1} - 2x_n + x_{n+1}] \tag{23}$$

Let's take a guess

$$x_n = B e^{ipn} e^{i\omega t} \tag{24}$$

That the time dependence is exponential follows from linearity – we always guess this. Here we are also guessing that since our numerical solution looks like sine functions the n dependence should be oscillatory.

Plugging our guess in we get

$$-\omega^2 e^{ipn} = \left[e^{ip(n-1)} - 2e^{ipn} + e^{ip(n+1)}\right]\omega_0^2 \tag{25}$$

Dividing both sides by $-e^{ipn}$ gives

$$\omega^2 = [2 - e^{-ip} - e^{ip}]\omega_0^2 \tag{26}$$

$$= [2 - 2\cos(p)]\omega_0^2 \tag{27}$$

Thus, we have found solutions for any B as long as ω and p are related by

$$\boxed{\omega(p) \equiv \pm\sqrt{2(1-\cos(p))}\omega_0} \tag{28}$$

this is a type of **dispersion relation**. We will come back to dispersion relations later (once we have talked about dispersion).

To fix p, we need to use the boundary conditions, that is, the equations of motion for the end masses. The dispersion relationship doesn't tell us the sign of p. In fact, both p and $-p$ lead to the same ω so we can add solutions with p an $-p$ and still have a solution. Considering that the numerical solutions looked like sine curves, let's guess

$$x_n = B\sin(pn)e^{i\omega t} \tag{29}$$

Now, mass 1 satisfies Eq. (15), $m\frac{d^2x_1}{dt^2} = -2kx_1 + kx_2$. Plugging in our guess gives

$$-B\omega^2 \sin p = B\omega_0^2[-2\sin p + \sin 2p] = -B\omega_0^2 \sin p[2 - 2\cos p] \tag{30}$$

where $\sin(2x) = 2\sin(x)\cos(x)$ was used. Using Eq. (27) we then find that our guess works. You should make sure that you agree that Eq. (29) satisfies both Eq. (15) and Eq. (18) at this point.

Finally, we need to use the equation for mass N. It satisfies $m\frac{d^2x_N}{dt^2} = kx_{N-1} - 2kx_N$ as in Eq. (19). So

$$-B\omega^2 \sin Np = B\omega_0^2[\sin(N-1)p - 2\sin Np] \tag{31}$$

Substituting Eq. (27) on the left-hand side, canceling the $-2\sin Np$ terms, and then expanding $\sin(Np - p)$ using $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$ gives

$$2\sin Np\cos p = \sin(N-1)p = \sin Np\cos p - \sin p\cos Np \tag{32}$$

Thus

$$0 = \sin Np\cos p + \sin p\cos Np = \sin((N+1)p) \tag{33}$$

This equation is only satisfied if

$$\boxed{p = \frac{\pi}{N+1}j, \quad j = 1, 2, 3, \cdots} \tag{34}$$

This are the same eigenvalues we found by guessing after Eq. (22). We have now derived rigorously that the equations of motion for masses 1 and N correspond to boundary conditions where we hold masses $n = 0$ and $n = N+1$ fixed.

Thus the normal mode frequencies are

$$\boxed{\omega^2 = 2\left(1 - \cos\frac{\pi}{N+1}j\right)\omega_0^2, \quad j = 1, 2, 3 \cdots N} \tag{35}$$

and the solutions are $x_n = B\sin(pn)e^{\pm i\omega t}$. Let's check that this is the right answer for $N = 3$. We find

$$j=1, \quad \omega^2 = 2\Big(1-\cos\frac{\pi}{4}\Big)\omega_0^2 = \big(2-\sqrt{2}\big)\omega_0^2 \tag{36}$$

$$j=2, \quad \omega^2 = 2\Big(1-\cos\frac{\pi}{2}\Big)\omega_0^2 = 2\omega_0^2 \tag{37}$$

$$j=3, \quad \omega^2 = 2\left(1-\cos\frac{3\pi}{4}\right)\omega_0^2 = \big(2+\sqrt{2}\big)\omega_0^2 \tag{38}$$

These are exactly the frequencies we found in Eqs. (10) to (12).

For large N, the lowest frequencies have $j \ll N$ thus using

$$\cos(x) = 1 - \frac{1}{2}x^2 + \cdots, \quad x \ll 1 \tag{39}$$

we find

$$\omega^2 = 2\Big(1-\cos\frac{\pi}{N+1}j\Big)\omega_0^2 = \Big(\frac{\pi}{N+1}j\Big)^2\omega_0^2 = p^2\omega_0^2 \tag{40}$$

That is

- For a large number of modes, $\omega = \omega_0 p$: the frequency is proportional to the wavenumber

In other words, the dispersion relation becomes linear:

$$\omega(p) = p\omega_0 \tag{41}$$

This linearity will be important when we discuss dispersion.

4.4 Summary

In summary, we found the following solution for a large N number of masses connected by springs. For each integer $j = 1, 2, 3, \cdots$ there is a single normal mode solution. The position of mass n during the oscillation of normal mode j is given by

$$x_n^{(j)}(t) = \sin\left(\frac{\pi j}{N+1}n\right)\cos(\omega_j t + \phi_j) \tag{42}$$

We have chosen to write the solutions in manifestly real form (meaning we use sines and cosines rather than exponentials). The phase ϕ_j is arbitrary. The frequencies are given by

$$\omega_j = \omega_0\sqrt{2\Big(1-\cos\frac{\pi}{N+1}j\Big)}, \tag{43}$$

The normal mode solutions are periodic with frequencies ω_j.

For small j

$$\omega_j \approx \frac{\pi j}{N+1}\omega_0 \tag{44}$$

An arbitrary solution can be written as a sum over normal modes as

$$(\vec{x})_n(t) = \sum_j a_j \sin\left(\frac{\pi j}{N+1}n\right)\cos(\omega_j t + \phi_j) \tag{45}$$

for some real constants a_j and ϕ_j

These solutions all satisfy the boundary conditions $(\vec{x}_j)_0(t) = (\vec{x}_j)_{N+1}(t) = 0$. It is straightforward to work out the general solution for other boundary conditions.

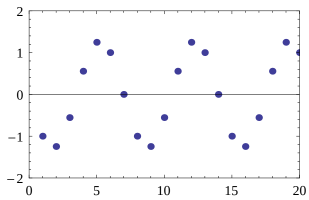
5 Continuum limit

We will now take the limit $N \to \infty$. This will turn our discrete problem into a continuous problem, and our differences into derivatives.

With N masses, we called the displacement of each mass from its equilibrium point x_n. Since all the springs have the same constant, at equilibrium, all the masses are a distance Δx apart. Let us define a function $A(\Delta x,\, t)$ as the **amplitude** of the displacement from equilibrium at a point x. So,

$$A(n\Delta x, t) = x_n(t) \tag{46}$$

Thus, plots like this



(47)

are plots of $A(x,\, t)$. To be clear, these displacements are still longitudinal (in the direction of the springs), but we are drawing $A(x,\, t)$ in the transverse direction. $A(x,\, t)$ so far only has values at discrete points given by $x = n\Delta x$. Its value at those points is $x_n(t)$.

In terms of $A(x, t)$, the equations of motion for the coupled system, Eq. (18)

$$\frac{d^2 x_n}{dt^2} = \frac{k}{m}[x_{n-1} - 2x_n + x_{n+1}] \tag{48}$$

become

$$\frac{\partial^2}{\partial t^2} A(n\Delta x, t) = \frac{k}{m}\,[A((n+1)\Delta x, t) - 2A(n\Delta x, t) + A((n-1)\Delta x, t)] \tag{49}$$

$$= \frac{k}{m}\Delta x\left[\frac{A(n\Delta x + \Delta x, t) - A(n\Delta x, t)}{\Delta x} - \frac{A(n\Delta x, t) - A(n\Delta x - \Delta x, t)}{\Delta x}\right] \tag{50}$$

Writing $x = n\Delta x$ this becomes

$$\frac{\partial^2}{\partial t^2} A(x, t) = \frac{k}{m}\Delta x\left[\frac{A(x + \Delta x, t) - A(x, t)}{\Delta x} - \frac{A(x, t) - A(x - \Delta x, t)}{\Delta x}\right] \tag{51}$$

Starting to look like calculus...

As $\Delta x \to 0$, this becomes

$$\frac{d^2}{dt^2} A(x, t) = \frac{k}{\mu}\left[\frac{\partial A(x, t)}{\partial x} - \frac{\partial A(x - \Delta x, t)}{\partial x}\right] \tag{52}$$

where $\mu = \frac{m}{\Delta x}$ is the mass per unit length or **mass density**. We also define $E = k\Delta x$ as the **elastic modulus** to get

$$\frac{d^2}{dt^2} A(x, t) = \frac{E}{\mu}\frac{1}{\Delta x}\left[\frac{\partial A(x, t)}{\partial x} - \frac{\partial A(x - \Delta x, t)}{\partial x}\right] \tag{53}$$

Now take $\Delta x \to 0$ turns the first derivatives into second derivatives. Writing

$$v \equiv \sqrt{\frac{E}{\mu}} \tag{54}$$

and we find

$$\boxed{\frac{\partial^2}{\partial t^2} A(x, t) = v^2 \frac{\partial^2}{\partial x^2} A(x, t)} \tag{55}$$

This is the **wave equation**.

6 Solving the wave equation

The wave equation is linear, so we can solve it with exponentials. Writing

$$A(x, t) = e^{i\omega t} e^{ikx} \tag{56}$$

we get

$$\omega^2 = v^2 k^2 \tag{57}$$

So

$$\omega(k) = v|k| \tag{58}$$

This is a linear dispersion relation. Since we have taken $N \to \infty$, all modes have $j \ll N$, thus, the linearity of the dispersion relation is consistent with what we found for finite N,

Since the wave equation (without damping) has only second derivatives, its easy to see that the general solution for a fixed frequency ω is

$$\boxed{A_k(x,t) = a_k\cos(\,kx\,)\cos(\omega t) + b_k\sin(kx)\cos(\omega t) + c_k\cos(kx\,)\sin(\omega t) + d_k\sin(kx)\sin(\omega t)} \tag{59}$$

exactly as in the discrete case. The only difference is that now

$$\omega(k) = vk \tag{60}$$

instead of the more complicated $\omega(p) = \sqrt{2(1-\cos(p))}\omega_0$ we found before.

Note that in the continuum case k has dimensions of $\frac{1}{\text{length}}$. We call k the wavenumber, which is equal to $\frac{2\pi}{\text{wevelength}}$. In the discrete case p was dimensionless. Note that this k has nothing to do with the spring constant – we just use the same letter "k" for both.

Which of a_k, b_k, c_k or d_k vanish depends on boundary conditions. Let's consider some interesting cases. First, we note that one solution is

$$A(x,t) = \cos(kx\,)\cos(\omega t) - \sin(kx)\sin(\omega t) = \cos(kx - \omega t) \tag{61}$$

$$= \cos\Big(\frac{\omega}{v}(x - vt)\Big) \tag{62}$$

This solution has the property that $A(x, t + \Delta t) = A(x - v\Delta t, t)$ meaning that the amplitude at x in the future is given by the amplitude at position to the left at current time. In other words, the curve is moving to the right. This is a **right-moving traveling wave**.

More generally, we note that for any function $f(z)$ the amplitude

$$A(x,t) = f(x - vt) \tag{63}$$

will satisfy the wave equation. Indeed, it is easy to check that

$$\frac{\partial^2}{\partial t^2} f(x - vt) = v^2 f''(x - vt) \tag{64}$$

$$\frac{\partial^2}{\partial x^2} f(x - vt) = f''(x - vt) \tag{65}$$

So that

$$\left[\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right] f(x - vt) = 0 \tag{66}$$

Thus $A(x, t) = f(x - vt)$ is a general right-moving traveling wave. In general it will not be associated with a fixed frequency. However, since any solution can be written as a sum over normal modes, any traveling wave can be written as a sum over solutions of fixed frequency. How this is done is known as the Fourier decomposition, which we study next time.

Waves of the form

$$A(x,t) = f(x + vt) \tag{67}$$

are also solutions for any f. These are **left-moving** traveling waves.

Another solution of fixed frequency is

$$A(x,t) = \cos(kx - \omega t) + \cos(kx - \omega t) = 2\cos(kx)\cos(\omega t) \tag{68}$$

This solution has the property that the amplitudes at any two points x_1 and x_2 always have the same ratio at any time

$$\frac{A(x_1,t)}{A(x_2,t)} = \frac{2\cos(kx_1)}{2\cos(kx_2)} \tag{69}$$

These are **standing waves**.

We see that

- Whether a traveling wave or a standing wave is produced depends on initial conditions.
- Standing waves are the sum of a left-moving and right-moving wave.